9 The power of calculus

Try this worksheet after you have completed Exercise 9X.

Volume of a torus and other figures

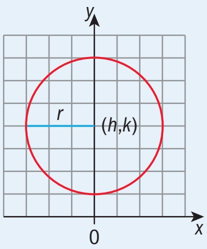
EXAMPLE

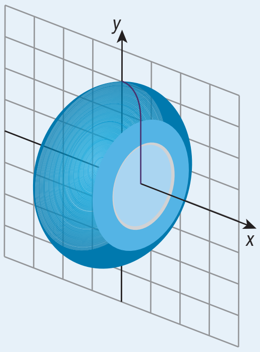
Find the volume of a torus that is obtained by rotating a circle with center at (h, k) and radius r where k, $r > 0$ and $k > r$, about the x-axis for 2π.

Answer

A circle with center at (h, k) and radius r is an implicitly defined function with the equation $(x - h)^2 + (y - k)^2 = r^2$.
Let $h = 0 \Rightarrow x^2 + (y - k)^2 = r^2$

Horizontal translation has no impact on the volume of the torus.





Notice that if we do a vertical translation the volume of the torus would change, since there would be a longer or shorter radius from the center of the tube to the center of the torus.

$x^2 + (y - k)^2 = r^2 \Rightarrow (y - k)^2 = r^2 - x^2$

$\Rightarrow y - k = \pm\sqrt{r^2 - x^2}$

$\Rightarrow y = k \pm \sqrt{r^2 - x^2}$

Express y as a function of x so that we can proceed with the volume of revolution.

$$\Rightarrow \begin{cases} y_1 = k + \sqrt{r^2 - x^2} \\ y_2 = k - \sqrt{r^2 - x^2} \end{cases}$$

y_1 is an upper curve and y_2 is an lower curve.

$$V = \pi\int_{-r}^{r}\left((y_1)^2 - (y_1)^2\right)\mathrm{d}x = 2\pi\int_{0}^{r}\left((y_1)^2 - (y_1)^2\right)\mathrm{d}x$$

Use the symmetrical property of the circle to simplify the boundaries of the integration.

$$= 2\pi\int_0^r\left(\left(k + \sqrt{r^2 - x^2}\right)^2 - \left(k - \sqrt{r^2 - x^2}\right)^2\right)\mathrm{d}x$$

Apply the difference of two squares formula.

$$= 2\pi\int_0^r\left(2\sqrt{r^2 - x^2}\ 2k\right)\mathrm{d}x$$

Simplify the integral.

$$= 8k\pi\int_0^r\sqrt{r^2 - x^2}\,\mathrm{d}x$$

Let $x = r\sin\theta \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta} = r\cos\theta \quad x_1 = 0 \Rightarrow \theta_1 = 0$

Apply a special substitution from 9.6 and change the boundaries of integration.

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$\sqrt{r^2 - x^2} = r\cos\theta \quad x_2 = r \Rightarrow \theta_2 = \frac{\pi}{2}$	*Simplify*
$= 8k\pi \int_0^{\frac{\pi}{2}} r\cos\theta\, r\cos\theta\, \mathrm{d}\theta$	
$= 8kr^2\pi \int_0^{\frac{\pi}{2}} \cos^2\theta\, \mathrm{d}\theta$	
$= 8kr^2\pi \int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\mathrm{d}\theta$	*Apply a double angle formula of cosine*
$= 8kr^2\pi \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\frac{\pi}{2}}$	
$= 8kr^2\pi \left(\left(\frac{\pi}{4} + \frac{\sin\pi}{4}\right) - \left(0 + \frac{\sin 0}{4}\right)\right)$	*Apply Fundamental Theory of Calculus*
$= 8kr^2\pi \cdot \frac{\pi}{4} = 2kr^2\pi^2$	*Simplify*

Volume of a torus and other figures

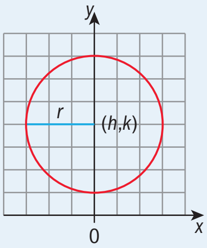
EXAMPLE

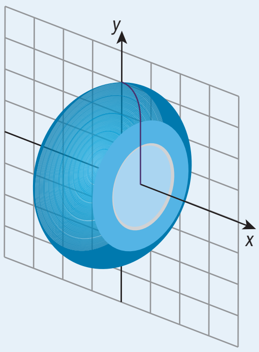
Find the volume of a torus that is obtained by rotating a circle with center at (h, k) and radius r where k, $r > 0$ and $k > r$, about the x-axis for 2π.

Answer

A circle with center at (h, k) and radius r is an implicitly defined function with the equation $(x - h)^2 + (y - k)^2 = r^2$.
Let $h = 0 \Rightarrow x^2 + (y - k)^2 = r^2$

Horizontal translation has no impact on the volume of such a torus.





Notice that if we do a vertical translation the torus would change volume since there will be a longer or shorter radius from the center of the tube to the center of the torus.

$x^2 + (y - k)^2 = r^2 \Rightarrow (y - k)^2 = r^2 - x^2$

$\Rightarrow y - k = \pm\sqrt{r^2 - x^2}$

$\Rightarrow y = k \pm \sqrt{r^2 - x^2}$

$\Rightarrow \begin{cases} y_1 = k + \sqrt{r^2 - x^2} \\ y_2 = k - \sqrt{r^2 - x^2} \end{cases}$

Express y as a function of x so that we can proceed with the volume of revolution.

y_1 is an upper curve and y_2 is a lower curve.

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$x^2 + (y - k)^2 = r^2 \Rightarrow (y - k)^2 = r^2 - x^2$ $\Rightarrow y - k = \pm\sqrt{r^2 - x^2}$ $\Rightarrow y = k \pm \sqrt{r^2 - x^2}$	*Express y as a function of x so that we can proceed with the volume of revolution.*
$\Rightarrow \begin{cases} y_1 = k + \sqrt{r^2 - x^2} \\ y_2 = k - \sqrt{r^2 - x^2} \end{cases}$	*y_1 is an upper curve and y_2 is a lower curve.*
$V = \pi\int_{-r}^{r}\left((y_1)^2 - (y_2)^2\right)\mathrm{d}x = 2\pi\int_{0}^{r}\left((y_1)^2 - (y_1)^2\right)\mathrm{d}x$	*Use the symmetrical property of the circle to simplify the boundaries of the integration.*
$= 2\pi\int_0^r\left(\left(k + \sqrt{r^2 - x^2}\right)^2 - \left(k - \sqrt{r^2 - x^2}\right)^2\right)\mathrm{d}x$	*Apply the difference of two squares formula.*
$= 2\pi\int_0^r = \left(2\sqrt{r^2 - x^2}\ 2k\right)\mathrm{d}x$ $= 8k\pi\int_0^r\sqrt{r^2 - x^2}\,\mathrm{d}x$	*Simplify the integral.*
Let $x = r\sin\theta \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta} = r\cos\theta \quad x_1 = 0 \Rightarrow \theta_1 = 0$ $\sqrt{r^2 - x^2} = r\cos\theta \quad x_2 = r \Rightarrow \theta_2 = \frac{\pi}{2}$	*Apply a special substitution from 9.6 and change the boundaries of integration.*
$= 8k\pi\int_0^{\frac{\pi}{2}} r\cos\theta \cdot r\cos\theta\,\mathrm{d}\theta$ $= 8kr^2\pi\int_0^{\frac{\pi}{2}}\cos^2\theta\,\mathrm{d}\theta$	*Simplify.*
$= 8kr^2\pi\int_0^{\frac{\pi}{2}}\frac{1 + \cos 2\theta}{2}\mathrm{d}\theta$	*Apply a double angle formula of cosine.*
$= 8kr^2\pi\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\frac{\pi}{2}}$	*Apply FTC.*
$= 8kr^2\pi\left(\left(\frac{\pi}{4} + \frac{\sin\pi}{4}\right) - \left(0 + \frac{\sin 0}{4}\right)\right)$ $= 8kr^2\pi \cdot \frac{\pi}{4} = 2kr^2\pi^2$	*Simplify*

Exercise

1. Use volume of revolution to find the volume of a torus obtained by the rotating circle $(x - 4)^2 + (y + 3)^2 = 4$ about the x-axis.
2. Use volume of revolution to find the volume of a torus obtained by the rotating circle $(x - 4)^2 + (y + 3)^2 = 4$ about the y-axis.
3. Use volume of revolution to find the volume of a sphere obtained by the rotating upper part of the circle $x^2 + y^2 = 9$ about the x-axis for 2π.
4. Find the volume of a solid obtained by the rotating upper part of the ellipse $4x^2 + 9y^2 = 36$ about the x-axis for 2π.
5. Find the volume of a solid obtained by rotating the right part of the ellipse $4x^2 + 9y^2 = 36$ about the y-axis for 2π.
6. Find the volume of a solid obtained by rotating the upper part of the ellipse $4(x - 2)^2 + 9y^2 = 36$ about the x-axis for 2π.

Chapter 9 extension worked solutions

1 Let's use the tip on the horizontal translation.

$x^2 + (y + 3)^2 = 4 \Rightarrow (y + 3)^2 = 4 - x^2$

$\Rightarrow y + 3 = \pm\sqrt{4 - x^2}$

$\Rightarrow y = -3 \pm \sqrt{4 - x^2}$

$\Rightarrow \begin{cases} y_1 = -3 + \sqrt{4 - x^2} \\ y_2 = -3 - \sqrt{4 - x^2} \end{cases}$

y_2 is a curve further to the x-axis and y_1 is a curve closer to the x-axis.

$V = \pi\int_{-2}^{2}\left((y_2)^2 - (y_1)^2\right)dx = 2\pi\int_0^2\left((y_2)^2 - (y_1)^2\right)dx$

Use the symmetrical property of the circle to simplify the boundaries of the integration.

$= 2\pi\int_0^2\left(\left(-3 - \sqrt{4 - x^2}\right)^2 - \left(-3 + \sqrt{4 - x^2}\right)^2\right)dx$

Apply the difference of two squares formula.

$= 2\pi\int_0^2\left(2\sqrt{4 - x^2}\cdot 6\right)dx$

$= 24\pi\int_0^2\sqrt{4 - x^2}\,dx$

Simplify the integral.

Let $x = 2\sin\theta \Rightarrow \frac{dx}{d\theta} = 2\cos\theta \quad x_1 = 0 \Rightarrow \theta_1 = 0$

$\sqrt{4 - x^2} = 2\cos\theta \quad x_2 = 2 \Rightarrow \theta_2 = \frac{\pi}{2}$

Apply a special substitution from 9.6 and change the boundaries of integration.

$= 24\pi\int_0^{\frac{\pi}{2}} 2\cos\theta\cdot 2\cos\theta\,d\theta$

Simplify

$= 96\pi\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta$

$= 96\pi\int_0^{\frac{\pi}{2}}\frac{1 + \cos 2\theta}{2}d\theta$

Apply a double angle formula of cosine.

$= 96\pi\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\frac{\pi}{2}}$

Apply FTC

$= 96\pi\left(\left(\frac{\pi}{4} + \frac{\sin\pi}{4}\right) - \left(0 + \frac{\sin 0}{4}\right)\right)$

$= 96\pi\cdot\frac{\pi}{4} = 24\pi^2$

Simplify

2 In this case the vertical translation will not influence the volume.

$(x - 4)^2 + y^2 = 4 \Rightarrow (x - 4)^2 = 4 - y^2$

$\Rightarrow x - 4 = \pm\sqrt{4 - y^2}$

$\Rightarrow x = 4 \pm \sqrt{4 - y^2}$

$\Rightarrow \begin{cases} x_1 = 4 + \sqrt{4 - y^2} \\ x_2 = 4 - \sqrt{4 - y^2} \end{cases}$

x_1 is a curve further to the y-axis and x_2 is a curve closer to the y-axis.

$V = \pi\int_{-2}^{2}\left((x_1)^2 - (x_2)^2\right)dy = 2\pi\int_0^2\left((x_1)^2 - (x_2)^2\right)dy$

Use the symmetrical property of the circle to simplify the boundaries of the integration.

$= 2\pi\int_0^2\left(\left(4 + \sqrt{4 - y^2}\right)^2 - \left(4 - \sqrt{4 - y^2}\right)^2\right)dy$

Apply the difference of two squares formula.

$= 2\pi\int_0^2\left(2\sqrt{4 - y^2}\cdot 8\right)dy$

$= 32\pi\int_0^2\sqrt{4 - y^2}\,dy$

Simplify the integral.

Let $y = 2\sin\theta \Rightarrow \frac{dy}{d\theta} = 2\cos\theta \quad y_1 = 0 \Rightarrow \theta_1 = 0$

$\sqrt{4-y^2} = 2\cos\theta \quad y_2 = 2 \Rightarrow \theta_2 = \frac{\pi}{2}$

Apply a special substitution from 9.6 and change the boundaries of integration.

$$= 32\pi\int_0^{\frac{\pi}{2}} 2\cos\theta\, 2\cos\theta\, d\theta$$

Simplify

$$= 128\pi\int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta$$

Apply a double angle formula of cosine.

$$= 128\pi\int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\, d\theta$$

Apply FTC

$$= 128\pi\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\frac{\pi}{2}}$$

Simplify

$$= 128\pi\left(\left(\frac{\pi}{4} + \frac{\sin\pi}{4}\right) - \left(0 + \frac{\sin 0}{4}\right)\right)$$

$$= 128\pi\cdot\frac{\pi}{4} = 32\pi^2$$

3 Let's take the upper part of the circle.

Use the symmetrical property of the circle to simplify the boundaries of the integration.

$y^2 = 9 - x^2 \Rightarrow y = \sqrt{9-x^2}$

$$V = \pi\int_{-3}^{3}\left(\sqrt{9-x^2}\right)^2 dx = 2\pi\,(9-x^2)\,dx$$

Apply FTC

$$= 2\pi\left[9x - \frac{x^3}{3}\right]_0^3$$

Simplify

$$= 2\pi((27-9)-(0-0))$$

$$= 36\pi$$

4 Let's take the upper part of the ellipse.

Use the symmetrical property of the ellipse to simplify the boundaries of the integration.

$y^2 = 4 - \frac{4}{9}x^2 \Rightarrow y = \sqrt{4 - \frac{4}{9}x^2}$

$$V = \pi\int_{-3}^{3}\left(\sqrt{4-\frac{4}{9}x^2}\right)^2 dx = 2\pi\int_{-3}^{3}\left(4 - \frac{4}{9}x^2\right)dx$$

Apply FTC

$$= 2\pi\left[4x - \frac{4x^3}{27}\right]_0^3$$

Simplify

$$= 2\pi((12-4)-(0-0)) = 16\pi$$

5 Let's take the right part of the ellipse.

Use the symmetrical property of the ellipse to simplify the boundaries of the integration.

$x^2 = 9 - \frac{9}{4}y^2 \Rightarrow x = \sqrt{9 - \frac{9}{4}y^2}$

$$V = \pi\int_{-2}^{2}\left(\sqrt{9-\frac{9}{4}y^2}\right)^2 dy = 2\pi\int_{0}^{2}\left(9 - \frac{9}{4}y^2\right)dy$$

Apply FTC

$$= 2\pi\left[9y - \frac{3y^3}{4}\right]_0^2$$

Simplify

$$= 2\pi\left((18-6)-(0-0)\right)$$

$$= 24\pi$$

6 Since we rotate about the x-axis the horizontal translation will not influence the volume, therefore the result is the same as in question 4, $V = 16\pi$.