1 Mathematics as the science of patterns

Try this worksheet after you have completed Chapter 1.

Exercise

1 **a** A sequence of numbers is given by $u_n = \frac{1}{n} - \frac{1}{n+1}$.

i Simplify the expression for u_n in terms of n and write down the first six terms of the sequence.

ii Find $\prod_{r=1}^{n} u_r$ in terms of n.

b A sequence of numbers is defined as $u_1 = 3$, $u_n = 3 + u_{n-1}$ $n \geq 2$, $n \in \mathbb{Z}$.

i Write the first six terms of the sequence.

ii Find $S_n = \sum_{r=1}^{n} u_r$ and $P_n = \prod u_r$ in terms of n.

iii Hence show that $P_n \times S_n = \frac{9}{2} n(n+1)!$

2 **a** Prove using mathematical induction that $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$

b Given that $a_n = 2^{n-1}$ and $b_n = n(n-1)$, evaluate the following:

i $\sum_{r=1}^{n} a_r$ and $\sum_{r=1}^{n} b_r$

ii $\prod_{r=1}^{n} a_r$ and $\prod_{r=1}^{n} b_r$

iii $\dfrac{\prod_{r=1}^{n} a_r}{\left(\sum_{r=1}^{n} a_r\right)+1}$ and $\dfrac{\prod_{r=1}^{n} b_r}{\sum_{r=1}^{n} b_r}$

3 Let $S_n = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + \ldots + \frac{n-1}{n!}$, $n \in \mathbb{Z}^+$.

a Show that $S_1 = \frac{1}{2}$, $S_2 = \frac{5}{6}$ and $S_3 = \frac{23}{24}$.

b Find S_4, S_5 and S_6.

c Use your results to **a** and **b** to make a conjecture about S_n.

d Prove your conjecture using mathematical induction.

4 A geometric sequence has common ratio $\frac{1+c}{1-c}$ $c \in \mathbb{R}$. Show that for this sequence to converge $c < 0$. S_1 is the sum of a convergent geometric sequence with first term 1 and common ratio $\frac{1+c_1}{1-c_1}$ and S_2 is the sum of a convergent geometric sequence with first term 1 and ratio $\frac{1+c_2}{1-c_2}$. Show that $S_1 - S_2 = \frac{c_1 - c_2}{2c_1c_2}$.

5 Prove by mathematical induction that $\sum_{r=1}^{n} r^3 = \frac{1}{4} n^2 (n+1)^2$.

Let $S = 1^3 + 3^3 + 5^3 + \ldots + (2n+1)^3$ show that $S = (n+1)^2 (2n^2 + 4n + 1)$.

6 In how many ways can seven girls and five boys wait in a line if no two boys stand next to each other?

Notation

When dealing with sequences we define

$\sum_{r=1}^{n} a_r = a_1 + a_2 + a_3 + \ldots + a_n$, that is the sum of the first n terms of the sequence in Sigma (or sum) notation and

$\prod_{r=1}^{n} a_r = a_1 \times a_2 \times a_3 \times \ldots \times a_n$, that is the product of the first n terms of the sequence in product notation.

7 A club consists of six men and seven women.

- **a** In how many ways can a committee of five club members be formed?
- **b** In how many ways can a committee of three men and two women be formed?
- **c** In how many ways can we select a committee of five members that has at most one man?
- **d** Two married couples Mr and Mrs Park and Mr and Mrs Kim are members of the club. In how many different ways can the club members sit around a table given that Mr Park and Mr Kim sit next to their wives?

8 You are given the digits 2, 3, 7 and 8.

- **a** How many different numbers can be formed taking one, two, three and four digits if each digit appears only once in any number?
- **b** How many different numbers can be formed taking one, two, three and four digits if digits may appear more than once in any number
- **c** How many of the numbers found in part **a** are even and smaller than 500?
- **d** How many of the numbers found in part **b** are odd and greater than 500?

Chapter 1 extension worked solutions

Exercise

1 a i $u_n = \frac{1}{n} - \frac{1}{n+1} = \frac{n+1-n}{n(n+1)} = \frac{1}{n(n+1)}$ The first six terms are $\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \frac{1}{30}, \frac{1}{42}$.

ii $\prod_{r=1}^{n} u_r = \left(\frac{1}{1\times 2}\right)\times\left(\frac{1}{2\times 3}\right)\times\left(\frac{1}{3\times 4}\right)\times\ldots\times\left(\frac{1}{n\times n+1}\right)$

$= \frac{1}{(1\times 2\times 3\times\ldots\times n)(2\times 3\times 4\times\ldots\times(n+1))} = \frac{1}{n!(n+1)!}$

b i The first six terms are 3, 6, 9, 12, 15, 18.

This is an arithmetic sequence so

$S_n = \frac{n}{2}(3+3n) = \frac{3n}{2}(1+n)$

$P_n = 3\times 6\times 9\times 12\times\ldots\times 3n = 3(1\times 2\times 3\times 4\times\ldots\times n) = 3n!$

$S_nP_n = 3n!\times\frac{3n}{2}(1+n) = \frac{9}{2}n(n+1)!$

$P_nS_n = 3n!\times\frac{3n}{2}(1+n) = \frac{9}{2}n(n+1)!$

2 Let $P(n): \sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$

When $n = 1$ LHS = 1

RHS $= \frac{1}{6}(2)(3) = 1$

So $P(1)$ is true

Assume that $P(k)$ is true, i.e. $\sum_{r=1}^{k} r^2 = \frac{k}{6}(k+1)(2k+1)$.

Required to show that $P(k+1)$ is true i.e. $\sum_{r=1}^{k+1} r^2 = \frac{(k+1)}{6}(k+2)(2k+3)$.

When $n = k+1$

$$\text{LHS} = \sum_{r=1}^{k} r^2 + (k+1)^2 = \frac{k}{6}(k+1)(2k+1) + (k+1)^2$$
$$= \frac{(k+1)}{6}\left[k(2k+1) + 6(k+1)\right]$$
$$= \frac{(k+1)}{6}(2k^2+7k+6) = \frac{(k+1)}{6}(k+2)(2k+3)$$

Since $P(1)$ is true and we have shown that if $P(k)$ is true $P(k+1)$ is also true, it follows using the principle of mathematical induction that $P(n)$ is true for all $n \in \mathbb{Z}^+$.

a i $\sum_{r=1}^{n} a_r = 1+2+4+8+\ldots+2^{n-1}$ which is a geometric series

$\sum_{r=1}^{n} a_r = \frac{1(1-2^n)}{1-2} = 2^{n-1}$

$\sum_{r=1}^{n} b_r = \sum_{r=1}^{n} n(n+1) = \sum_{r=1}^{n} n^2 + n = \sum_{r=1}^{n} n^2 + \sum_{r=1}^{n} n$

$= \frac{n}{6}(n+1)(2n+1) + \frac{n}{2}(n+1)$

$= \frac{(n+1)}{6}\left[n(2n+1)+3n\right] = \frac{(n+1)}{6}(2n^2+n+3n) = \frac{n(n+1)(n+2)}{3}$

ii $\prod_{r=1}^{n} a_r = 1\times 2\times 2^2\times 2^3\times\ldots\times 2^{n-1} = 2^{(1+2+3+\ldots+(n-1))} = 2^{\frac{n(n-1)}{2}}$

$\prod_{r=1}^{n} b_r = (1\times 2)\times(2\times 3)\times(3\times 4)\times\ldots\times(n\times(n+1))$

$= (1\times 2\times 3\times\ldots\times n)(2\times 3\times 4\times\ldots\times(n+1)) = n!(n+1)!$

iii $$\frac{\prod_{r=1}^{n} a_r}{\left(\sum_{r=1}^{n} a_r\right)} = \frac{2^{\frac{n(n-1)}{2}}}{(2^n-1)+1} = \frac{2^{\frac{n(n-1)}{2}}}{2^n} = 2^{\frac{n(n-1)}{2}-n} = 2^{\frac{n(n-3)}{2}}$$

$$\frac{\prod_{r=1}^{n} b_r}{\sum_{r=1}^{n} b_r} = \frac{n!(n+1)!}{\frac{n(n+1)(n+2)}{3}} = \frac{3n!(n+1)!}{n(n+1)(n+2)} = \frac{3n!(n-1)!}{(n+2)}$$

3 **a** When $n = 1$ $S_1 = \frac{1}{2!} = \frac{1}{2}$

When $n = 2$ $S_2 = \frac{1}{2!} + \frac{2}{3!} = \frac{1}{2} + \frac{2}{6} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$

When $n = 3$ $S_3 = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} = \frac{1}{2} + \frac{2}{6} + \frac{3}{24} = \frac{12}{24} + \frac{8}{24} + \frac{3}{24} = \frac{23}{24}$

b When $n = 4$ $S_4 = \frac{23}{24} + \frac{4}{5!} = \frac{115}{120} + \frac{4}{120} = \frac{119}{120}$

When $n = 5$ $S_5 = \frac{119}{120} + \frac{5}{6!} = \frac{714}{720} + \frac{5}{720} = \frac{719}{720}$

When $n = 6$ $S_6 = \frac{719}{720} + \frac{6}{7!} = \frac{5033}{5040} + \frac{6}{5040} = \frac{5039}{5040}$

c $S_n = \frac{(n+1)!-1}{(n+1)!}$

d Let $S_n = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + \ldots + \frac{n-1}{n!}$, $n \in \mathbb{Z}^+$.

Let $P(n): \sum_{r=1}^{n} \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + \ldots + \frac{n-1}{n!} = \frac{(n+1)!-1}{(n+1)!}$

When $n = 1$ $\text{LHS} = \frac{1}{2}$

$\text{RHS} = \frac{2!-1}{2!} = \frac{1}{2}$

So $P(1)$ is true

Assume that $P(k)$ is true, i.e. $\sum_{r=1}^{k} \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + \ldots + \frac{k-1}{k!} = \frac{(k+1)!-1}{(k+1)!}$.

Required to show that $P(k + 1)$ is true i.e. $\sum_{r=1}^{k+1} \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + \ldots + \frac{k}{(k+1)!} = \frac{(k+2)!-1}{(k+2)!}$.

When $n = k + 1$

$\text{LHS} = \sum_{r=1}^{k+1} \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + \ldots + \frac{k}{(k+1)!} =$

Since $P(1)$ is true and we have shown that if $P(k)$ is true $P(k + 1)$ is also true, it follows using the principle of mathematical induction that $P(n)$ is true for all $n \in \mathbb{Z}^+$.

4 For convergence $\left|\frac{1+c}{1-c}\right| < 1$

When $c > 0$, $\frac{1+c}{1-c} > 1$ and so the series will not converge.

When $-1 < c < 0$, let $c = -k$, $k > 0$.

$r = \frac{1+c}{1-c} = \frac{1-k}{1+k}$ and since $k > 0$ it follows that $0 > r > 1$ hence the series will converge.

When $c < -1$ it follows that $1 + c < 1 - c \Rightarrow -1 < r < 0$ and so the sum converges.

$$S_1 = \frac{1}{1-\left(\frac{1+c}{1-c}\right)} = \frac{1}{\left(\frac{(1-c_1)-(1+c)}{1-c_1}\right)} = -\frac{1-c_1}{2c_1}$$

Similarly $S_2 = -\frac{1-c_2}{2c_2}$

$$S_1 - S_2 = -\left(\frac{1-c_1}{2c_1}\right) - \left(\frac{1-c_2}{2c_2}\right) = \frac{c_1(1-c_2) - c_2(1-c_1)}{2c_1c_2} = \frac{c_1 - c_2}{2c_1c_2}$$

5 Consider the statement $P(n)$: $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

When $n = 1$

$\text{LHS} = 1^3 = 1$

$\text{RHS} = \frac{1}{4}(1^2)(2^2) = 1$

$\text{LHS} = \text{RHS}$

so $P(1)$ is true.

Assume $P(k)$ is true i.e. $\sum_{r=1}^{k} r^3 = \frac{1}{4}k^2(k+1)^2$

When $n = k + 1$

$\text{LHS} = \sum_{r=1}^{k} r^3 + (k+1)^3 = \frac{1}{4}k^2(k+1)^2 + (k+1)^3 = \frac{(k+1)^2}{4}(k^2 + 4k + 4) = \frac{(k+1)^2(k+2)^2}{4}$

$\text{RHS} = \frac{1}{4}(k+1)^2 + (k+1+1)^2 = \frac{(k+1)^2(k+2)^2}{4}$

$\text{LHS} = \text{RHS}$

So $P(k+1)$ is true if $P(k)$ is true. But since it was shown that $P(1)$ is true, it follows by the principle of mathematical induction that $P(n)$ is true for all $n \in \mathbb{Z}^+$

Let $R = \sum_{r=1}^{2n+1} r^3 = \frac{(2n+1)^2(2n+2)^2}{4} = (2n+1)^2(n+1)^2$

$S = R - (2^3 + 4^3 + 6^3 + 8^3 + \ldots + (2n)^3) = R - 2^3(1^3 + 2^3 + 3^3 + \ldots + n^3)$

$S = (2n+1)^2(n+1)^2 - 8\left(\frac{n^2(n+1)^2}{4}\right) = (n+1)^2\left((2n+1)^2 - 2n^2\right) = (n+1)^2(2n^2 + 4n + 1)$

6 There are seven girls and five boys. The number of ways of arranging seven girls is 7! No two boys are not allowed to stand next to each other so we can solve the problem by creating spaces which can be filled by boys between the girls as follows.

_ G _ G _ G _ G _ G _ G _ G _

As can be seen the boys can now fill out eight possible spaces, so the number of arrangement of boys is given by $P_5^8 = \frac{8!}{(8-5)!} = \frac{8!}{3!}$

But each of these arrangements can be combined with 7! arrangements for the girls.

So the number of ways of arranging the girls and boys so that no two boys stand together is given by

$\frac{8!}{3!} \times 7! = 33868800$

7 **a** Since order is not important we are dealing with combinations.
We need to choose 5 out of 13

$$\binom{13}{5} = 1287$$

b $\binom{6}{3}\binom{7}{2} = 420$

c Two possible committees: one man and four women; or no men and five women

$\binom{6}{1}\binom{7}{4} + \binom{7}{5} = 210 + 21 = 231$

d Each couple is now taken as one unit. We now have a total of 11 to be seated around a table. But each couple can sit in two ways (spouse on the left or on the right).

$$10! \times 2! \times 2! = 14515200$$

8 a Number of ways of forming 1 digit numbers is 4

Number of ways of forming 2 digit numbers is $4 \times 3 = 12$

Number of ways of forming 3 digit numbers is $4 \times 3 \times 2 = 24$

Number of ways of forming 4 digit numbers is $4 \times 3 \times 2 \times 1 = 24$

In total: $4 + 12 + 24 + 24 = 64$

b Number of ways of forming 1 digit numbers is 4

Number of ways of forming 2 digit numbers is $4^2 = 16$

Number of ways of forming 3 digit numbers is $4^3 = 64$

Number of ways of forming 4 digit numbers is $4^4 = 256$

In total: $4 + 16 + 64 + 256 = 340$

c We cannot have any 4 digit numbers

Even one digit numbers less than $500 \rightarrow 2$

$$\left.\begin{array}{l}\text{Even 2 digit numbers:}\\ \text{Two ways of choosing the units digit}\\ \text{Even three digit numbers:}\\ \text{Three ways of choosing the tens digit}\end{array}\right\} 2 \times 3 = 6$$

Two ways of choosing the units digit i.e. 2 or 8

If 2 is in the units then there is only one way of choosing the hundreds digit leaving two ways of choosing the tens digit.

If 8 is the units digit we have two ways of choosing the hundreds digit and another two ways of choosing the tens digit.

In all 2 + 4 ways of forming 3 digit even numbers less than 500.

In total: $2 + 6 + 6 = 14$

d We can only have 3 digit and 4 digit numbers

Odd 3 digit numbers bigger than 500:

2 ways of choosing the hundreds digit

4 ways of choosing the tens digit

2 ways of choosing the units digit

Odd 4 digit numbers greater than 500:

4 ways of choosing the thousands digit

4 ways of choosing the hundreds digit

4 ways of choosing the tens digit

2 ways of choosing the units digit

In total: $2 \times 4 \times 2 + 4^3 \times 2 = 144$.