8 Ancient mathematics and modern methods

Try this worksheet after you have completed section 8.10

Catenaries and other curves

The photograph shows the Gateway Arch, in St. Louis, Missouri. It is made of stainless steel and the construction was completed in 1965. Historically it symbolizes the gateway for pioneers as they made their way to the West.

Mathematically the shape of the arch is important because of the stability it provides. The shape is called a catenary and stability is maximized because the weight of the arch acts along its legs directly into the ground.

A general catenary curve is the shape taken by a chain suspended at its ends and is described by the equation $y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$.

Exercise 1

1 Use your GDC to draw the graph of a catenary curve for:

a $a = 5$.

b $a = -5$.

c $a = \frac{1}{5}$.

d What is the equation for a catenary curve that has the same shape as

$y = -\left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right)$ and opens in the same direction but has a y-intercept of 4?

2 **a** Using your GDC, on the same axes draw the graphs of

$f(x) = e^x$, $g(x) = e^{-x}$ and $h(x) = \frac{f(x)+g(x)}{2}$ and $k(x) = \frac{f(x)-g(x)}{2}$.

b Show that $h(x)$ is an even function.

c Show that $k(x)$ is an odd function.

3 The functions $h(x)$ and $k(x)$ are called hyperbolic functions defined as follows:

$h(x) = \cos h(x) = \frac{1}{2}(e^x + e^{-x})$ called the hyperbolic cosine function

$k(x) = \sin h(x) = \frac{1}{2}(e^x - e^{-x})$ called the hyperbolic sine function.

Prove that

a $\cosh x + \sinh x = e^x$ and $\cosh x - \sinh x = e^{-x}$. Hence show that $(\cosh x)^2 - (\sinh x)^2 = 1$

b $\sinh 2x = 2\sinh x \cosh x$

c $\cosh 2x = \cosh^2 x + \sinh^2 x$

d Given that $\tanh x = \frac{\sin h x}{\cos h x}$, show that $\tanh 2x = \frac{2\tan h x}{1+\tan h^2 x}$.

4 The hypocycloid is a curve defined by the parametric equations
$x(t) = \cos^3 t$, $y(t) = \sin^3 t$, $0 \le t \le 2\pi$.

a Use the GDC in parametric mode to graph the hypocycloid.

b Use the parametric equations above to write the Cartesian equation of the hypocycloid.

c Graph the Cartesian equation obtained and compare to the graph obtained in **a**.

5 An epicycloid is a curve given by the parametric equations

$$\left.\begin{aligned} x(t) &= a\left(\frac{n+1}{n}\right)\cos t - \frac{a}{n}\cos(n+1)t \\ y(t) &= a\left(\frac{n+1}{n}\right)\sin t - \frac{a}{n}\sin(n+1)t \end{aligned}\right\} n \in \mathbb{Z}$$

a The cardiod is a special epicycloid given by putting $n = 1$. Use your GDC to graph the cardiods for $a \in \{1, -1, 3\}$.

b When $n = 2$ the epicycloid has two cusps and is called a Nephroid.

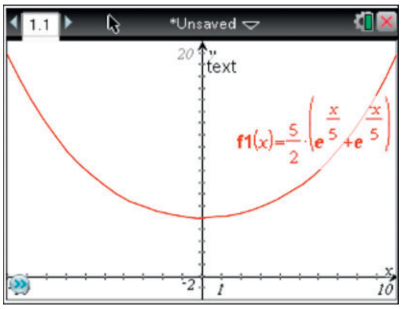
Use your GDC to graph nephroids for $a \in \{1, -1, 3\}$.

c Using the results above, what would you expect the epicycloids for $n = 3$, 4 and 5 to look like?

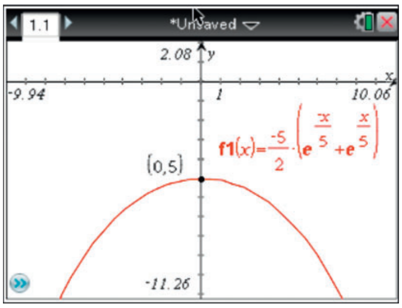
Verify by graphing these functions on your GDC.

Chapter 8 extension worked solutions

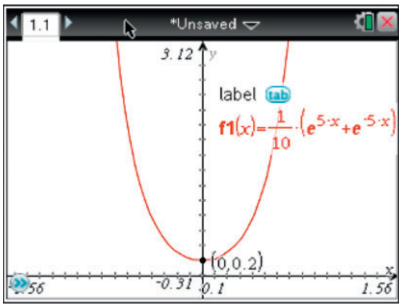
1 a



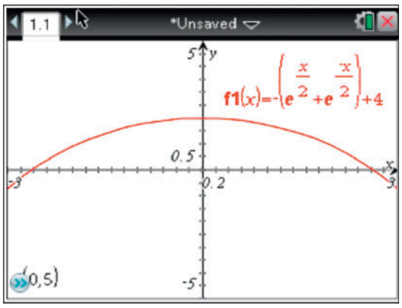
b



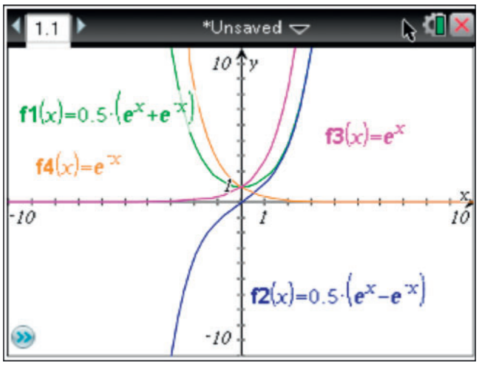
c



d Equation is given by $y = -\left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right) + 4$



2 a



b $h(-x) = \dfrac{f(-x) + g(-x)}{2} = \dfrac{e^{-x} + e^{x}}{2} = h(x)$, so $h(x)$ is an even function.

c $k(-x) = \dfrac{f(-x) - g(-x)}{2} = \dfrac{e^{-x} - e^{x}}{2} = -\dfrac{e^{x} - e^{-x}}{2} = -k(x)$, so $k(x)$ is an odd function.

3 **a** $\cosh x + \sinh x = \dfrac{e^x + e^{-x}}{2} + \dfrac{e^x - e^{-x}}{2} = \dfrac{2e^x}{2} = e^x$

$\cosh x - \sinh x = \dfrac{e^x + e^{-x}}{2} - \dfrac{e^x - e^{-x}}{2} = \dfrac{2e^{-x}}{2} = e^{-x}$

$\cosh^2 x - \sinh^2 x = (\cosh x + \sinh x)(\cosh x - \sinh x) = e^x e^{-x} = 1$

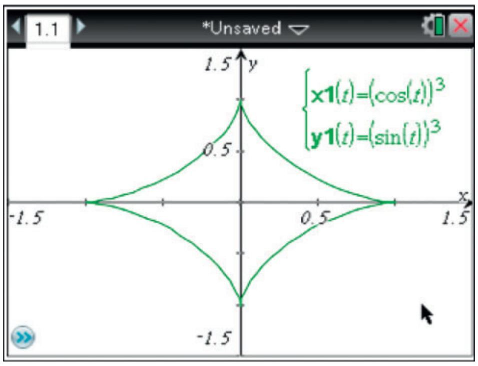
b $\sinh 2x = \dfrac{e^{2x} - e^{-2x}}{2} = \dfrac{(e^x)^2 - (e^{-x})^2}{2} = \dfrac{(e^x - e^{-x})(e^x + e^{-x})}{2} = 2\sinh x \cosh x$

c $\cosh^2 x + \sinh^2 x = \left(\dfrac{e^x + e^{-x}}{2}\right)^2 + \left(\dfrac{e^x - e^{-x}}{2}\right)^2$

$= \dfrac{e^{2x} + 2 + e^{-2x}}{4} + \dfrac{e^{2x} - 2 + e^{-2x}}{4} = \dfrac{e^{2x} + e^{-2x}}{2} = \cosh 2x$

d $\tanh 2x = \dfrac{\sinh 2x}{\cosh 2x} = \dfrac{2\sinh x \cosh x}{\cosh^2 x + \sinh^2 x} = \dfrac{\dfrac{2\sinh x \cosh x}{\cosh^2 x}}{\dfrac{\cosh^2 x}{\cosh^2 x} + \dfrac{\sinh^2 x}{\cosh^2 x}} = \dfrac{2\tanh x}{1 + \tanh^2 x}$

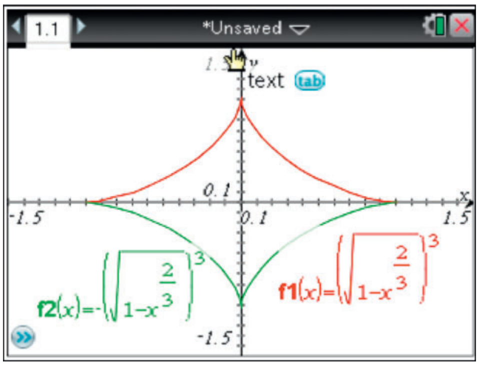
4 **a**



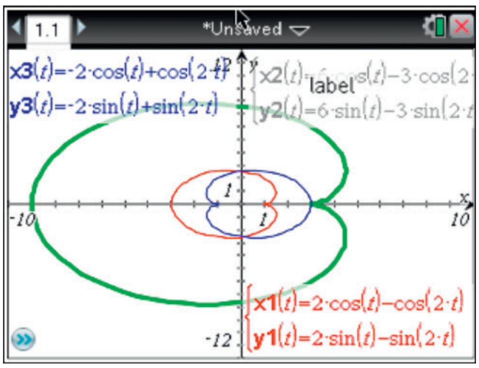
b $x = \cos^3 t \Rightarrow \cos t = x^{\frac{1}{3}}$

$y = -\left(\sqrt{1 - x^{\frac{2}{3}}}\right)^3 \quad y = \sin^3 t = (\sin t)^3 = \left(\sqrt{1 - \cos^2 t}\right)^3 = \left(\sqrt{1 - x^{\frac{2}{3}}}\right)^3$

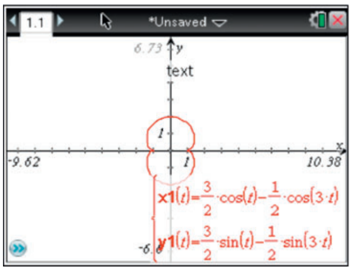
c In order to obtain the graph in **a**, one must also sketch the curve $y = -\left(\sqrt{1 - x^{\frac{2}{3}}}\right)^3$

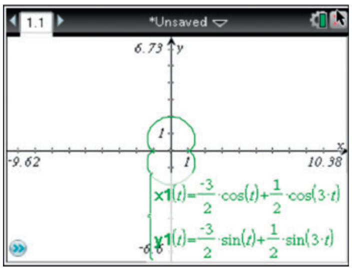


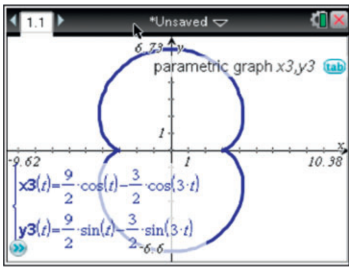
5 **a**



b







c From the graphs above it seems that the number of cusps is determined by n. It also seems that epicycloids are even functions when n is even. The value of a determines the width of the epicycloid. The larger a is, the further away the locus of points on the cycloid is from the origin. When a is negative the cycloid is reflected in the y-axis so when n is even this will not affect the graph.

Some examples to verify these statements:

$a = 1,\ n = 20$ $a = 10,\ n = 5$ $a = 7,\ n = 6$