6 Exploring randomness

Try this worksheet after you have completed section 6.5

To calculate the probability of a union of more events we use a general formula

$$P\left(\bigcup_{r=1}^{n} A_r\right)=\sum_{r=1}^{n} P(A_r)-\sum_{1\le i<j\le n} P(A_i\cap A_j)+\sum_{1\le i<j\le k<n} P(A_i\cap A_j\cap A_k)+\ldots+(-1)^{n+1}P\left(\bigcap_{r=1}^{n} A_r\right),\ n\in\mathbb{Z}^+.$$

This formula derived from the so called **inclusion-exclusion principle** and several mathematicians were working simultaneously on it. It is attributed to Abraham de Moivre, Daniel da Silva, Joseph Sylvester and Henri Poincare.

Use mathematical induction to prove the given formula for n events.

Proof:

We know that the formula works for one set, $n = 1$.
$P(A_1) = P(A_1)$
Assume that the formula works for k sets, $n = k$.

$$P\left(\bigcup_{r=1}^{k} A_r\right)=\sum_{r=1}^{k} P(A_r)-\sum_{1\le i<j\le k} P(A_i\cap A_j)+\ldots+(-1)^{n+1}P\left(\bigcap_{r=1}^{k} A_r\right),\ k\in\mathbb{Z}^+$$

Now we need to see whether the formula works for $n = k + 1$.

$$P\left(\bigcup_{r=1}^{k+1} A_r\right)=P\left(\left(\bigcup_{r=1}^{k} A_r\right)\cup A_{k+1}\right)$$

$$P\left(\bigcup_{r=1}^{k+1} A_r\right)=P\left(\bigcup_{r=1}^{k} A_r\right)+P(A_{k+1})-P\left(\left(\bigcup_{r=1}^{k} A_r\right)\cap A_{k+1}\right)$$

$$=P\left(\bigcup_{r=1}^{k} A_r\right)+P(A_{k+1})-P\left(\bigcup_{r=1}^{k}(A_r\cap A_{k+1})\right)$$

Notice that now we have two unions with k sets in each and by using the assumption we can rewrite the formula.

$$=\sum_{r=1}^{k} P(A_r)-\sum_{1\le i<j\le k} P(A_i\cap A_j)+\ldots+(-1)^{n+1}P\left(\bigcap_{r=1}^{k} A_r\right)+P(A_{k+1})$$

$$-\left(\sum_{r=1}^{k} P(A_r\cap A_{k+1})-\sum_{1\le i<j\le k} P(A_i\cap A_j\cap A_{k+1})+\ldots+(-1)^{n+1}P\left(\left(\bigcap_{r=1}^{k} A_r\right)\cap A_{k+1}\right)\right)$$

$$=\sum_{r=1}^{k+1} P(A_r)-\sum_{1\le i<j\le k+1} P(A_i\cap A_j)+\ldots+(-1)^{k+2}P\left(\bigcap_{r=1}^{k+1} A_r\right)$$

Since the statement was shown to be true for $n = 1$, and it was also proved that if the statement is true for $n = k$ it is also true for $n = k + 1$, it follows by the principle of mathematical induction that the statement is true for all positive integers n.

Exercise

1 Given the integers between 1 and 1000, find the probability that a randomly selected integer is divisible by any of the primes that are less than 10.

2 Five camels form a camel train or caravan and reach an oasis. After a day of rest they randomly form a new caravan. Find the probability that no camel will be directly behind the same camel it was before the rest in the oasis.

Chapter 6 extension worked solutions

1 Let's denote by:

A "All the integers less than 1000 that are divisible by 2."

B "All the integers less than 1000 that are divisible by 3."

C "All the integers less than 1000 that are divisible by 5."

D "All the integers less than 1000 that are divisible by 7."

$P(A \cup B \cup C \cup D) = P(A) + P(B) + P(C) + P(D)$

$-P\underbrace{(A \cap B)}_{\text{divisible by 6}} -P\underbrace{(A \cap C)}_{\text{divisible by 10}} -P\underbrace{(A \cap D)}_{\text{divisible by 14}} -P\underbrace{(B \cap C)}_{\text{divisible by 15}} -P\underbrace{(B \cap D)}_{\text{divisible by 21}} -P\underbrace{(C \cap D)}_{\text{divisible by 35}}$

$+P\underbrace{(A \cap B \cap C)}_{\text{divisible by 30}} +P\underbrace{(A \cap B \cap D)}_{\text{divisible by 42}} +P\underbrace{(A \cap C \cap D)}_{\text{divisible by 70}} +P\underbrace{(B \cap C \cap D)}_{\text{divisible by 105}} -P\underbrace{(A \cap B \cap C \cap D)}_{\text{divisible by 210}}$

$= \frac{1}{1000}\left(\left\lfloor\frac{1000}{2}\right\rfloor + \left\lfloor\frac{1000}{3}\right\rfloor + \left\lfloor\frac{1000}{5}\right\rfloor + \left\lfloor\frac{1000}{7}\right\rfloor - \left\lfloor\frac{1000}{6}\right\rfloor - \left\lfloor\frac{1000}{10}\right\rfloor - \left\lfloor\frac{1000}{14}\right\rfloor -\right.$

$\left.\left\lfloor\frac{1000}{15}\right\rfloor - \left\lfloor\frac{1000}{21}\right\rfloor - \left\lfloor\frac{1000}{35}\right\rfloor + \left\lfloor\frac{1000}{30}\right\rfloor + \left\lfloor\frac{1000}{42}\right\rfloor + \left\lfloor\frac{1000}{70}\right\rfloor + \left\lfloor\frac{1000}{105}\right\rfloor - \left\lfloor\frac{1000}{210}\right\rfloor\right)$

$= \frac{1}{1000}(1175 - 478 + 79 - 4) = 0.772$

2 In this question we are going to use complementary events. Let's denote the following events by:

A "The second camel is directly after the first camel."

B "The third camel is directly after the second camel."

C "The fourth camel is directly after the third camel."

D "The fifth camel is directly after the fourth camel."

$P(A' \cap B' \cap C' \cap D') = 1 - P(A \cup B \cup C \cup D) = 1 - P(A) - P(B) - P(C) - P(D)$

$+ P(A \cap B) + P(A \cap C) + P(A \cap D) + P(B \cap C) + P(B \cap D) + P(C \cap D)$

$- P(A \cap B \cap C) - P(A \cap B \cap D) - P(A \cap C \cap D) - P(B \cap C \cap D) + P(A \cap B \cap C \cap D)$

$= \frac{1}{5!}\left(5! - 4 \cdot 4! + 6 \cdot 3! - 4 \cdot 2! + 1\right) = \frac{53}{120}$