5 Aesthetics in mathematics

Try this worksheet after you have completed section 5.6

Natural logarithms and e

1 **a** Consider the series $S = 1 - x + x^2 - x^3 + \ldots$

Show that if $|x| < 1$, $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n-1}\left(\frac{x^n}{n}\right)$.

b It can be shown that the result obtained above is also true when $x = 1$.

Show that $\log_2 e - \log_4 e + \log_8 e - \log_{16} e + \ldots = 1$.

2 Assume that $f(x) = e^x$ can be written as an infinite series of ascending powers of x i.e.

$f(x) = e^x = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$

i Show that $a_0 = f(0) = 1$ and $a_1 = f'(0) = 1$.

ii Show that $f^n(0) = n!a_n$.

iii Write the expansion of $f(x) = e^x$.

iv Hence calculate e to six decimal places.

3 Show that the general term of the series $1 + \frac{3x}{1!} + \frac{5x^2}{2!} + \frac{7x^3}{3!} + \ldots$ can be written as $\frac{x^n}{n!} + 2x\left(\frac{x^{n-1}}{(n-1)!}\right)$.

Hence evaluate $1 + \frac{3}{1!} + \frac{5}{2!} + \frac{7}{3!} + \ldots$

(Use the result for the expansion of $f(x) = e^x$ obtained in question **2**).

4 Use the expansion for $f(x) = e^x$ obtained in question **2** to show that

a $e^{ix} + e^{-ix}$, where $i = \sqrt{-1}$, is real for all $x \in \mathbb{R}$.

b $e^{ix} - e^{-ix}$, where $i = \sqrt{-1}$, is imaginary for all $x \in \mathbb{R}$.

5 The equation $x^y = y^x$ is called a transcendental equation. A trivial solution to this equation would be any ordered pair (a, a). In this question we will look for other nontrivial real solutions to this equation.

a **i** Show that $(-2, -4)$ is a solution to the equation $x^y = y^x$. Write two other ordered pairs (a, b), where $a, b \in \mathbb{Z}$, which satisfy this equation.

ii Evaluate $x = \left(\frac{n+1}{n}\right)^{n+1}$ and $y = \left(\frac{n+1}{n}\right)^{n}$ when $n = 1$. Show that in this case $x^y = y^x$.

iii Use technology to evaluate $x = \left(\frac{n+1}{n}\right)^{n+1}$ and $y = \left(\frac{n+1}{n}\right)^{n}$ for $n \in \mathbb{Z}^+$ and verify that $x^y = y^x$ for these values of x and y.

iv Prove that $x = \left(\frac{n+1}{n}\right)^{n+1}$ and $y = \left(\frac{n+1}{n}\right)^{n}$ are solutions to the equation $x^y = y^x$ for $n \in \mathbb{Z}^+$.

b **i** Evaluate $x = k^{\frac{1}{k-1}}$ and $y = k^{\frac{k}{k-1}}$ when $k = 3$ and use your result to show that $(\sqrt{3},\ 3\sqrt{3})$ is a solution to the equation $x^y = y^x$.

ii Use technology to evaluate $x = k^{\frac{1}{k-1}}$ and $y = k^{\frac{k}{k-1}}$, $k \in \mathbb{R}$, $k \neq 1$, and to verify that $x^y = y^x$ for these values of x and y.

iii Use the substitution $y = kx$, $k \in \mathbb{R}$, $k \neq 1$, in the equation $x^y = y^x$ to obtain the parametric equations: $x = k^{\frac{1}{k-1}}$ and $y = k^{\frac{k}{k-1}}$.

Chapter 5 extension worked solutions

Exercise 1

1 **a** This is a geometric series with first term 1 and common ratio $-x$

Since we are given that $|x| < 1$, it follows that $S = \frac{1}{1+x}$.

Therefore $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$

Taking the definite integral of both sides between 0 and x we get:

$$\int_0^x \left(\frac{1}{1+x}\right) dx = \int_0^x \left(1 - x + x^2 - x^3 + \ldots\right) dx$$

$$\left[\ln(1+x)\right]_0^1 = \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots(-1)^n\left(\frac{x^n}{n}\right) + \ldots\right]_0^x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots(-1)^n\left(\frac{x^n}{n}\right) + \ldots$$

b $\log_2 e - \log_4 e + \log_8 e - \log_{16} e + \ldots$

$$= \frac{1}{\ln 2} - \frac{1}{\ln 4} + \frac{1}{\ln 8} - \frac{1}{\ln 16} + \ldots$$

$$= \frac{1}{\ln 2} - \frac{1}{2\ln 2} + \frac{1}{3\ln 2} - \frac{1}{4\ln 2} + \ldots$$

$$= \frac{1}{\ln 2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\right)$$

Substituting $x = 1$ in the expansion $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots(-1)^n\left(\frac{x^n}{n}\right) + \ldots$ we get

$$\ln 2 = \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\right)$$

Hence it follows $\log_2 e - \log_4 e + \log_8 e - \log_{16} e + \ldots = \frac{1}{\ln 2} \times \ln 2 = 1$

2 **a** $f(x) = e^x = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$

$\Rightarrow f(0) = e^0 = a_0$

$\therefore a_0 = 1$

$f'(x) = e^x = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots$

$\Rightarrow f'(0) = e^0 = a_1$

$\therefore a_1 = 1$

b $f''(x) = e^x = 2a_2 + (3 \times 2)a_3 x + (4 \times 3)x^2 + \ldots$

So it follows that $f''(0) = e^0 = 2a_2 \Rightarrow a_2 = \frac{1}{2}$

Differentiating again we obtain

$f'''(x) = e^x = 3!a_3 = (4 \times 3 \times 2)a_4 x + \ldots$

$\Rightarrow f'''(0) = e^0 = 3!a_3$

$\therefore a_3 = \frac{1}{3!}$

If we continue in this way we find that

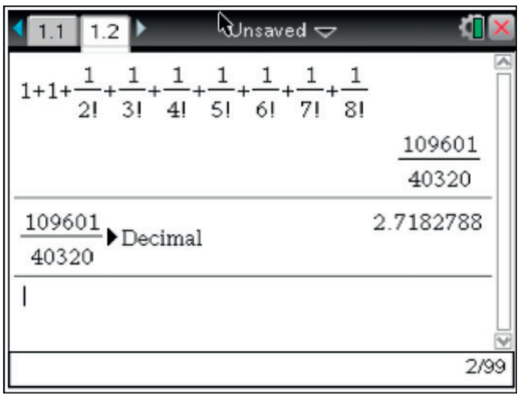
$f^n(0) = e^0 = n!a_n \Rightarrow a_n = \frac{1}{n!}$

c Using the results above we obtain

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$$

d Substituting $x = 1$ in this expansion we obtain

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \ldots = 2.718279$$



3 $1 + \frac{3x}{1!} + \frac{5x^2}{2!} + \frac{7x^3}{3!} + \ldots$

The general term of this sequence is $\frac{(2n+1)x^n}{n!} = \frac{2nx^n}{n!} + \frac{x^n}{n!} = 2x\frac{(x^{n-1})}{(n-1)!} + \frac{x^n}{n!}$

$$1 + \frac{3x}{1!} + \frac{5x^2}{2!} + \frac{7x^3}{3!} + \ldots = 1 + \sum_{n=1}^{\infty} 2x\frac{(x^{n-1})}{(n-1)!} + \sum_{n=1}^{\infty}\frac{x^n}{n!}$$

$$2x\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) + \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) = (1 + 2x)e^x$$

When $x = 1$ we obtain $1 + \frac{3}{1!} + \frac{5}{2!} + \frac{7}{3!} + \ldots = 3e$

4 $e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \ldots$

$e^{-ix} = 1 - ix - \frac{x^2}{2!} + \frac{ix^3}{3!} + \frac{x^4}{4!} - \frac{ix^5}{5!} - \ldots$

$e^{ix} + e^{-ix} = 2\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right)$ which is real and

$e^{ix} - e^{-ix} = 2i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} = \ldots\right)$ which is imaginary.

$i = \sqrt{-1}$
$i^2 = -1$
$i^3 = -i$
$i^4 = 1$
$i^5 = i$
And then the pattern repeats itself

5 a i $(-2)^{-4} = \frac{1}{(-2)^4} = \frac{1}{16}$

$(-4)^{-2} = \frac{1}{(-4)^2} = \frac{1}{16}$

Therefore $(-2, -4)$ is a solution to the equation $x^y = y^x$.

Another ordered pair would be $(-4, -2)$.

ii $x = \left(\frac{n+1}{n}\right)^{n+1} = 2^2 = 4$ $\quad y = \left(\frac{n+1}{n}\right)^n = 2^1 = 2$

$x^y = 4^2 = 16$ $\quad y^x = 2^4 = 16$

iii

	B	C	D	E
◆	=((a[]+1)/	=((a[]+1)/a	=b[]^c[]	=c[]^b[]
1	4	2	16	16
2	27/8	9/4	15.4389	15.4389
3	256/81	64/27	15.2969	15.2969
4	3125/10...	625/256	15.24	15.24

iv $x^y = \left(\left(\frac{n+1}{n}\right)^{n+1}\right)^{\left(\frac{n+1}{n}\right)^n} = \left(\frac{n+1}{n}\right)^{(n+1)\left(\frac{n+1}{n}\right)^n} = \left(\frac{n+1}{n}\right)^{\frac{(n+1)^{n+1}}{n^n}}$

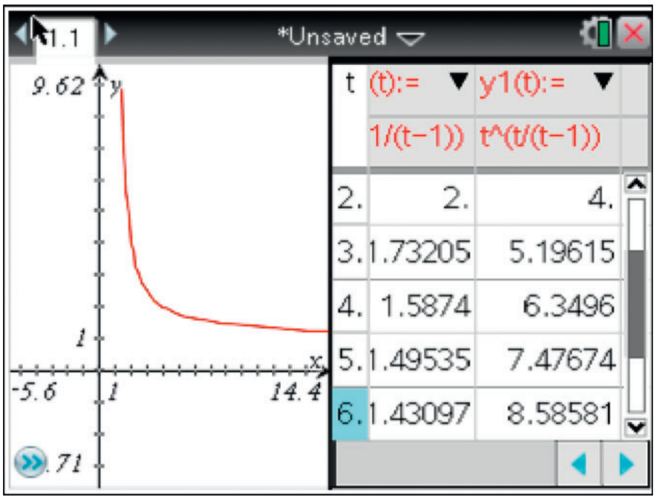
$y^x = \left(\left(\frac{n+1}{n}\right)^{n}\right)^{\left(\frac{n+1}{n}\right)^{n+1}} = \left(\frac{n+1}{n}\right)^{n\left(\frac{n+1}{n}\right)^{n+1}} = \left(\frac{n+1}{n}\right)^{\frac{(n+1)^{n+1}}{n^n}}$

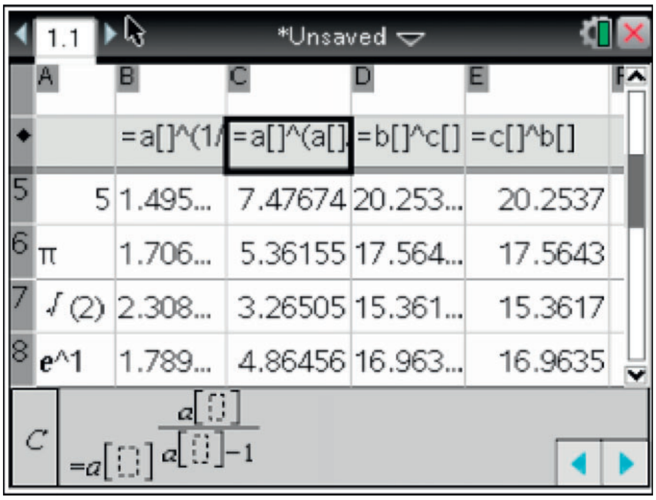
b i $x = k^{\frac{1}{k-1}} = 3^{\frac{1}{2}} = \sqrt{3}$

$y = k^{\frac{k}{k-1}} = 3^{\frac{3}{2}} = 3\sqrt{3}$

$x^y = \left(\sqrt{3}\right)^{3\sqrt{3}} = \left(\left(\sqrt{3}\right)^3\right)^{\sqrt{3}} = \left(3\sqrt{3}\right)^{\sqrt{3}} = y^x$

ii





iii $x^y = y^x$

Let $y = kx$, $k \in \mathbb{R}$, $k \neq 0$

$x, y \neq 0$ since this is a trivial case $x^0 = y^0 = 1$

$x^{kx} = (kx)^x$

Taking logs

$kx \ln x = x \ln(kx)$

$k \ln x = \ln k + \ln x$

$(k - 1) \ln x = \ln k$

$\ln x = \left(\frac{1}{k-1}\right) \ln k$

$\ln x = \ln k^{\left(\frac{1}{k-1}\right)}$

$\therefore x = k^{\left(\frac{1}{k-1}\right)}$

Since

$y = kx$

$y = k\left(k^{\left(\frac{1}{k-1}\right)}\right) = k^{1+\left(\frac{1}{k-1}\right)} = k^{\frac{k}{k-1}}$