11 Inspiration and formalism

Proof of results for 3-D vectors

Exercise 1

Try this section after you have studied the proof for 2-D vectors on page 590.

1 Use the geometric definition of the scalar product to establish the distributive property:

$\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$ for any vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.

Hence show that $(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{c} + \boldsymbol{b}) = \boldsymbol{a} \cdot \boldsymbol{c} + \boldsymbol{a} \cdot \boldsymbol{d} + \boldsymbol{b} \cdot \boldsymbol{c} + \boldsymbol{b} \cdot \boldsymbol{d}$ for any vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$.

First prove this for the case where the angles between the vectors are all acute. You need to consider other cases to obtain the general proof.

2 Given two vectors in the 3-D space $\boldsymbol{u} = u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k}$ and $\boldsymbol{v} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}$, use the distributive property to show that $\boldsymbol{u} \cdot \boldsymbol{v} = u_1v_1 + u_2v_2 + u_3v_3$.

3 Given two vectors in the 3-D space $\boldsymbol{u} = u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k}$ and $\boldsymbol{v} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}$, use the cosine rule to show that if $\boldsymbol{u} \cdot \boldsymbol{v} = |\boldsymbol{u}||\boldsymbol{v}|\cos\theta$ where θ is the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$, then $\boldsymbol{u} \cdot \boldsymbol{v} = u_1v_1 + u_2v_2 + u_3v_3$. Discuss the validity of this proof.

Proof of Lagrange's formula

Read this proof after you have completed Section 11.5

EXAMPLE 1

Prove that

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$$

Answer

Let $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$.

$$\text{LHS} = \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_2c_3 - b_3c_2 \\ b_3c_1 - b_1c_3 \\ b_1c_2 - b_2c_1 \end{pmatrix} = \begin{pmatrix} a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3) \\ a_3(b_2c_3 - b_3c_2) - a_1(b_1c_2 - b_2c_1) \\ a_1(b_3c_1 - b_1c_3) - a_2(b_2c_3 - b_3c_2) \end{pmatrix}$$

$$= \begin{pmatrix} a_2b_1c_2 - a_2b_2c_1 - a_3b_3c_1 + a_3b_1c_3 \\ a_3b_2c_3 - a_3b_3c_2 - a_1b_1c_2 + a_1b_2c_1 \\ a_1b_3c_1 - a_1b_1c_3 - a_2b_2c_3 + a_2b_3c_2 \end{pmatrix}$$

$$\text{RHS} = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c} = (a_1c_1 + a_2c_2 + a_3c_3)\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} - (a_1b_1 + a_2b_2 + a_3b_3)\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$= \begin{pmatrix} a_1c_1b_1 + a_2c_2b_1 + a_3c_3b_1 \\ a_1c_1b_2 + a_2c_2b_2 + a_3c_3b_2 \\ a_1c_1b_3 + a_2c_2b_3 + a_3c_3b_3 \end{pmatrix} - \begin{pmatrix} a_1b_1c_1 + a_2b_2c_1 + a_3b_3c_1 \\ a_1b_1c_2 + a_2b_2c_2 + a_3b_3c_2 \\ a_1b_1c_3 + a_2b_2c_3 + a_3b_3c_3 \end{pmatrix} = \begin{pmatrix} a_2c_2b_1 + a_3c_3b_1 - a_2b_2c_1 - a_3b_3c_1 \\ a_1c_1b_2 + a_3c_3b_2 - a_1b_1c_2 - a_3b_3c_2 \\ a_1c_1b_3 + a_2c_2b_3 - a_1b_1c_3 - a_2b_2c_3 \end{pmatrix}.$$

By inspection, we conclude that LHS = RHS   QED

Try these activities when you have completed the section on the angle between two lines

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Distance, equations and curves in the plane and in the space

Exercise 2

You could use dynamic geometry software to explore particular cases before attempting the general case.

1 Given the points $A(a_1, a_2)$ and $B(b_1, b_2)$, identify the locus of the point $P(x, y)$ defined by each of these conditions:

a $PA = AB$ **b** $PA = PB$ **c** $PA + PB = k$ where $k \in \mathbb{R}^+$.

In each case, use the definition $AB = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$ to deduce the Cartesian equations for the curve identified.

Conditions and regions in space

Exercise 3

You could use dynamic geometry software to explore particular cases before attempting the general case.

1 Given the points $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$, identify the locus of the points defined by these conditions:

a $PA = AB$ **b** $PA = PB$ **c** $PA + PB = k$ where $k \in \mathbb{R}^+$.

In each case, use the definition $AB = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}$ to deduce Cartesian equations for the curve identified.

2 Let $A(-1, 0, 3)$, $B(1, -2, 1)$ and $C(1, 1, -1)$.

a Find Cartesian equations for (AB) and (AC).

b Write down a Cartesian equation for the spherical surface S with center A that contains the point B.

c Hence determine the coordinates of the points D and E where S meets (AC).

d Show that $\overrightarrow{AE} + \overrightarrow{AB}$ and $\overrightarrow{AD} + \overrightarrow{AB}$ are orthogonal. Hence, explain how this shows that the bisectors of the angles between the lines (AB) and (AC) are perpendicular.

Investigation – more coefficient patterns

Consider the system $\begin{cases} a_1x + a_2y + a_3z = a_4 \\ a_5x + a_6y + a_7z = a_8 \\ a_9x + a_{10}y + a_{11}z = a_{12} \end{cases}$

Use a GDC to find the solutions of this system when

a $a_1, a_2, \ldots, a_{12}$ are consecutive multiples of an integer k

b $a_1, a_2, \ldots, a_{12}$ are consecutive terms of an arithmetic sequence.

Is there a general pattern? Write down a conjecture and prove it.

Try this investigation after you have completed the investigation of coefficient patterns on page 613.

New distance definitions and square circles

Try this activity after you have completed Chapter 11

Exercise 4

Given the point $A(a, b)$ and a positive real number r, we define the circle with center A and radius r as the locus of the point $P(x, y)$ such that $PA = r$

Draw this circle where

a distance $PA = |x - a| + |y - b|$

b distance $PA = \max(|x - a|, |y - b|)$.

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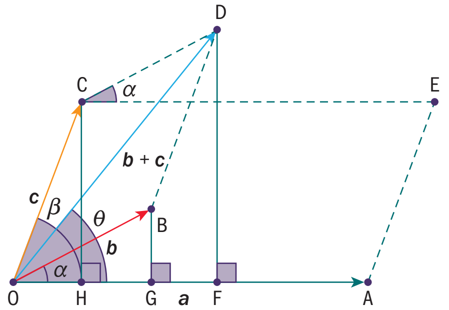
Chapter 11 extension worked solutions

Exercise 1

We are going to prove it first for the case where the angles between the vectors are all acute. You need to consider other cases to obtain the general proof.

Consider the diagram where the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are represented.



Let $\overrightarrow{\mathrm{OA}} = \overrightarrow{\mathrm{CE}} = \boldsymbol{a}$, $\overrightarrow{\mathrm{OB}} = \overrightarrow{\mathrm{CD}} = \boldsymbol{b}$ and $\overrightarrow{\mathrm{CD}} = \boldsymbol{c}$; let α be the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$, β be the angle between $\boldsymbol{a}$ and $\boldsymbol{c}$, and θ be the angle between $\boldsymbol{a}$ and $\boldsymbol{b} + \boldsymbol{c}$ (assume $\alpha < \beta$)

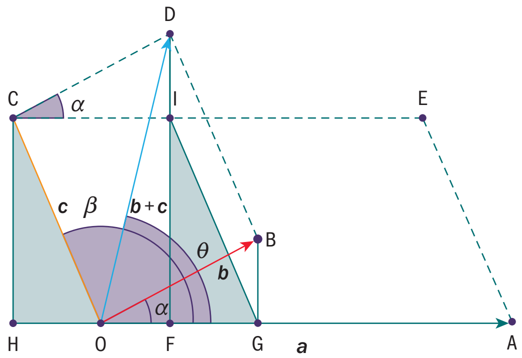
Using the geometric definition of scalar product, we have
$\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = |\boldsymbol{a}||\boldsymbol{b} + \boldsymbol{c}|\cos\theta = |\overrightarrow{\mathrm{OA}}||\overrightarrow{\mathrm{CF}}|$ where F is the foot of the perpendicular dropped from D on OA.

On the other hand,
$$\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c} = |\boldsymbol{a}||\boldsymbol{b}|\cos\alpha + |\boldsymbol{a}||\boldsymbol{c}|\cos\beta = |\boldsymbol{a}|\left(|\boldsymbol{b}|\cos\alpha + |\boldsymbol{c}|\cos\beta\right) = |\overrightarrow{\mathrm{OA}}|\left(\underbrace{|\overrightarrow{\mathrm{HF}}| + |\overrightarrow{\mathrm{OH}}|}_{|\overrightarrow{\mathrm{OF}}|}\right)$$

where G and H are the feet of the perpendiculars dropped from B and C on OA.

Therefore, $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$. QED.

The proof for the cases where one or more angles between the vectors are obtuse is similar. For example, if β is obtuse just consider the point I where [CE] meets [DF] and show that the triangles HOC and FGI are congruent and the result follows as before.



Now, as the scalar product is commutative, $(\boldsymbol{b} + \boldsymbol{c}) \cdot \boldsymbol{a} = \boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c} = \boldsymbol{b} \cdot \boldsymbol{a} + \boldsymbol{c} \cdot \boldsymbol{a}$.

To prove that $(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{c} + \boldsymbol{d}) = \boldsymbol{a} \cdot \boldsymbol{c} + \boldsymbol{a} \cdot \boldsymbol{d} + \boldsymbol{b} \cdot \boldsymbol{c} + \boldsymbol{b} \cdot \boldsymbol{d}$ we apply the distributive property repeatedly:

$$(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{c} + \boldsymbol{d}) = (\boldsymbol{a} + \boldsymbol{b}) \cdot \boldsymbol{c} + (\boldsymbol{a} + \boldsymbol{b}) \cdot \boldsymbol{d} = \boldsymbol{a} \cdot \boldsymbol{c} + \boldsymbol{b} \cdot \boldsymbol{c} + \boldsymbol{a} \cdot \boldsymbol{d} + \boldsymbol{b} \cdot \boldsymbol{d} = \boldsymbol{a} \cdot \boldsymbol{c} + \boldsymbol{a} \cdot \boldsymbol{d} + \boldsymbol{b} \cdot \boldsymbol{c} + \boldsymbol{b} \cdot \boldsymbol{d}$$

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2 $\boldsymbol{u}\cdot\boldsymbol{v} = (u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k})\cdot(v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k})$.

Using the distributive property

$$\boldsymbol{u}\cdot\boldsymbol{v} = (u_1\boldsymbol{i})\cdot(v_1\boldsymbol{i}) + (u_1\boldsymbol{i})\cdot(v_2\boldsymbol{j}) + (u_1\boldsymbol{i})\cdot(v_3\boldsymbol{k}) + (u_2\boldsymbol{j})\cdot(v_1\boldsymbol{i}) + (u_2\boldsymbol{j})\cdot(v_2\boldsymbol{j}) + (u_2\boldsymbol{j})\cdot(v_3\boldsymbol{k}) + (u_3\boldsymbol{k})\cdot(v_1\boldsymbol{i}) + (u_3\boldsymbol{k})\cdot(v_2\boldsymbol{j}) + (u_3\boldsymbol{k})\cdot(v_3\boldsymbol{k})$$

As $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ are mutually orthogonal, $\boldsymbol{i}\cdot\boldsymbol{j} = \boldsymbol{j}\cdot\boldsymbol{i} = \boldsymbol{i}\cdot\boldsymbol{k} = \boldsymbol{k}\cdot\boldsymbol{i} = \boldsymbol{j}\cdot\boldsymbol{k} = \boldsymbol{k}\cdot\boldsymbol{j} = 0$.

Using the property $(\alpha\boldsymbol{u})\cdot(\beta\boldsymbol{v}) = (\alpha\beta)\boldsymbol{u}\cdot\boldsymbol{v}$, we get

$\boldsymbol{u}\cdot\boldsymbol{v} = (u_1v_1)(\boldsymbol{i}\cdot\boldsymbol{i}) + (u_2v_2)(\boldsymbol{j}\cdot\boldsymbol{j}) + (u_3v_3)(\boldsymbol{k}\cdot\boldsymbol{k})$ as all the other terms are null.

As $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ are unit vectors, $\boldsymbol{i}\cdot\boldsymbol{i} = \boldsymbol{j}\cdot\boldsymbol{j} = \boldsymbol{k}\cdot\boldsymbol{k} = 1$

We conclude that $\boldsymbol{u}\cdot\boldsymbol{v} = u_1v_1 + u_2v_2 + u_3v_3$. QED

3 Using the cosine rule that we proved geometrically in chapter 8, we get

$|\boldsymbol{u}-\boldsymbol{v}|^2 = |\boldsymbol{u}|^2 + |\boldsymbol{v}|^2 - 2|\boldsymbol{u}||\boldsymbol{v}|\cos\theta.$

As $\boldsymbol{u}-\mathbf{v} = (u_1 - v_1)\boldsymbol{i} + (u_2 - v_2)\boldsymbol{j} + (u_3 - v_3)\boldsymbol{k}$, $|\boldsymbol{u}-\boldsymbol{v}| = (u_1-v_1)^2 + (u_2-v_2)^2 + (u_3-v_3)^2$

$|\boldsymbol{u}|^2 = u_1^2 + u_2^2 + u_3^2$ and $|\boldsymbol{v}|^2 = v_1^2 + v_2^2 + v_3^2$, we can simplify the identity above to obtain

$-2u_1v_1 - 2u_2v_2 - 2u_3v_3 = -2|\boldsymbol{u}||\boldsymbol{v}|\cos\theta \Rightarrow |\boldsymbol{u}||\boldsymbol{v}|\cos\theta = u_1v_1 + u_2v_2 + u_3v_3$

So, if $\boldsymbol{u}\cdot\boldsymbol{v} = |\boldsymbol{u}||\boldsymbol{v}|\cos\theta$ then $\boldsymbol{u}\cdot\boldsymbol{v} = u_1v_1 + u_2v_2 + u_3v_3$.

This proof uses $|\boldsymbol{u}|^2 = u_1^2 + u_2^2 + u_3^2$, which in fact results from the algebraic definition of scalar product. So, this is an example of circular reasoning.

Exercise 2

1 a Circle with center A and radius AB; $(x-a_1)^2 + (y-a_2)^2 = \underbrace{(a_1-b_1)^2 + (a_2-b_2)^2}_{r^2}$

b Perpendicular bisector of [AB]; $y = mx + c$ where $m = \frac{a_1-b_1}{b_2-a_2}$ and $c = \frac{b_1^2+b_2^2-a_1^2-a_2^2}{2b_2-2a_2}$.

c Ellipse with loci A and B $\sqrt{(x-a_1)^2 + (y-a_2)^2} + \sqrt{(x-b_1)^2 + (y-b_2)^2} = k$.

To obtain a simplified equation we need to choose the coordinates axes wisely. If the x-axis is chosen to be (AB) and the origin the midpoint of [AB],

the equation becomes much simpler: $\frac{4x^2}{k^2} + \frac{4y^2}{k^2-4c^2} = 1$, where c is the distance from each point A and B to the origin.

Exercise 3

1 a Spherical surface with center A and radius AB;

$$(x-a_1)^2 + (y-a_2)^2 + (z-a_3)^2 = \underbrace{(a_1-b_1)^2 + (a_2-b_2)^2 + (a_3-b_3)^2}_{r^2}$$

b Plane perpendicular to [AB] that bisects [AB]; $Ax + By + Cz = D$ where $\begin{pmatrix} A \\ B \\ C \end{pmatrix}$ is parallel to $\overrightarrow{AB}$ and $D = A\times\frac{a_1+b_1}{2} + B\times\frac{a_2+b_2}{2} + C\times\frac{a_3+b_3}{2}$

c Ellipsoid with loci A and B; an equation of this surface is

$\sqrt{(x-a_1)^2+(y-a_2)^2+(z-a_3)^2} + \sqrt{(x-b_1)^2+(y-b_2)^2+(z-b_3)^2} = k$ but you can obtain a simplified equation if you choose the coordinates axes wisely. If the x-axis is chosen to be the line (AB) and the origin the midpoint of (AB), the equation of this surface is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

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2 a $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

$\therefore$ line (AB): $\frac{x+1}{1} = \frac{y-0}{-1} = \frac{z-3}{-1}$

or

$x + 1 = -y = 3 - z$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix}$

$\therefore$ line (AC): $\frac{x+1}{2} = \frac{y-0}{1} = \frac{z-3}{-4}$

b $AB = |\overrightarrow{AB}| = \sqrt{2^2 + (-2)^2 + (-2)^2} = \sqrt{12} = r$

S: $(x+1)^2 + y^2 + (z-3)^2 = 12$

c To find the intersection points of S and AC we need to solve the system

$$\begin{cases} (x+1)^2 + y^2 + (z-3)^2 = 12 & (1) \\ \frac{x+1}{2} = y = \frac{z-3}{-4} & (2) \end{cases}$$

Using eq. (2) we have $x = 2y - 1$ and $z = -4y + 3$

Substitute in eq. (1) to obtain

$(2y - 1 + 1)^2 + y^2 + (-4y + 3 - 3)^2 = 12$

$4y^2 + y^2 + 16y^2 = 12 \Rightarrow y = \pm\frac{2}{\sqrt{7}}$

$\therefore D\left(\frac{4}{\sqrt{7}} - 1, \frac{2}{\sqrt{7}}, -\frac{8}{\sqrt{7}} + 3\right)$ and $E\left(-\frac{4}{\sqrt{7}} - 1, -\frac{2}{\sqrt{7}}, \frac{8}{\sqrt{7}} + 3\right)$

d $\overrightarrow{AE} = \overrightarrow{OE} - \overrightarrow{OA} = \begin{pmatrix} -\frac{4}{\sqrt{7}} - 1 \\ -\frac{2}{\sqrt{7}} \\ \frac{8}{\sqrt{7}} + 3 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -\frac{4}{\sqrt{7}} \\ -\frac{2}{\sqrt{7}} \\ \frac{8}{\sqrt{7}} \end{pmatrix}$

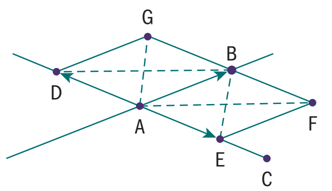
$\therefore \overrightarrow{AE} + \overrightarrow{AB} = \begin{pmatrix} -\frac{4}{\sqrt{7}} + 2 \\ -\frac{2}{\sqrt{7}} - 2 \\ \frac{8}{\sqrt{7}} - 2 \end{pmatrix}$

$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \begin{pmatrix} \frac{4}{\sqrt{7}} - 1 \\ \frac{2}{\sqrt{7}} \\ -\frac{8}{\sqrt{7}} + 3 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{4}{\sqrt{7}} \\ \frac{2}{\sqrt{7}} \\ -\frac{8}{\sqrt{7}} \end{pmatrix}$

$\overrightarrow{AD} + \overrightarrow{AB} = \begin{pmatrix} \frac{4}{\sqrt{7}} + 2 \\ \frac{2}{\sqrt{7}} - 2 \\ -\frac{8}{\sqrt{7}} - 2 \end{pmatrix}$

$(\overrightarrow{AE} + \overrightarrow{AB})(\overrightarrow{AD} + \overrightarrow{AB}) = 4 - \frac{16}{7} + 4 - \frac{4}{7} + 4 - \frac{64}{7} = 0$

$\therefore$ $\overrightarrow{AE} + \overrightarrow{AB}$ and $\overrightarrow{AD} + \overrightarrow{AB}$ are orthogonal



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Let F be such a point that

$\overrightarrow{AF} = \overrightarrow{AE} + \overrightarrow{AG}$

Then ABFE is a rhombus and

$\underbrace{\overrightarrow{AD} + \overrightarrow{AB}}_{\overrightarrow{EB}}$

and

$\underbrace{\overrightarrow{AE} + \overrightarrow{AB}}_{\overrightarrow{AF}}$

are its diagonals.

Let G be another point such that

$\overrightarrow{AG} = \overrightarrow{AD} + \overrightarrow{AB}$

Then ADGB is also a rhombus (congruent with ABFE) with diagonals

$\underbrace{\overrightarrow{AD} + \overrightarrow{AB}}_{\overrightarrow{AG}}$

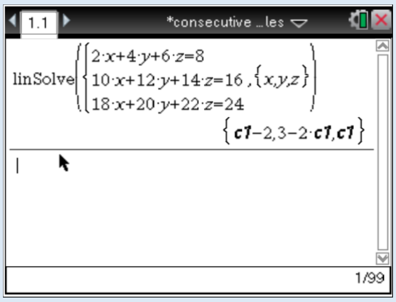
and

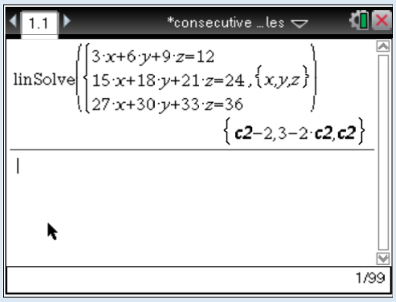
$\underbrace{\overrightarrow{AE} + \overrightarrow{AB}}_{\overrightarrow{DB}}$

As the diagonals of a rhombus bisect the angles of this polygon (and are mutually perpendicular), AG and AF bisect the angles between the lines AB and AC and are perpendicular.

Investigation – more coefficient patterns

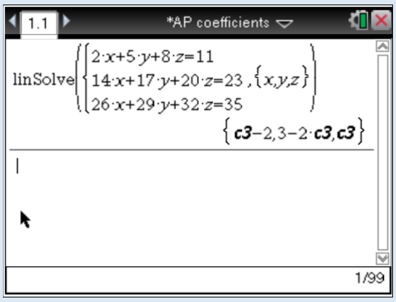
a Examples where the coefficients are consecutive multiples of 2 and 3.

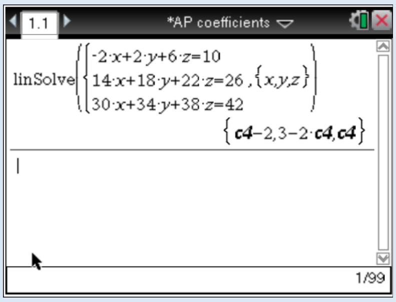




In both cases the systems have infinite solutions (i.e. the points on a line).

b Examples where the coefficients are consecutive terms of arithmetic sequences.





In all case the solutions are the coordinates of the points on the same straight line.

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Consider 12 consecutive terms of an arithmetic sequence:

$a, a+d, a+2d, \ldots, a+11d$

and the system

$$\begin{cases} ax + (a+d)y + (a+2d)z = a+3d & (1) \\ (a+4d)x + (a+5d)y + (a+6d)z = a+7d & (2) \\ (ax+8d)x + (a+9d)y + (a+10d)z = a+11d & (3) \end{cases}$$

As equation (2) is equal to ½(eq. (1)+eq.(2)), the system is equivalent to

$$\begin{cases} ax + (a+d)y + (a+2d)z = a+3d & (1) \\ (a+4d)x + (a+5d)y + (a+6d)z = a+7d & (2) \end{cases}$$

which defines a line L. To obtain a vector equal to L we can first simplify the system

$$\begin{cases} ax + (a+d)y + (a+2d)z = a+3d & (1) \\ 4dx + 4dy + 4dz = 4d & (\text{eq.}(2) - \text{eq.}(1)) \end{cases}$$

$$\Rightarrow \begin{cases} ax + (a+d)y + (a+2d)z = a+3d \\ x + y + z = 1 \end{cases}$$

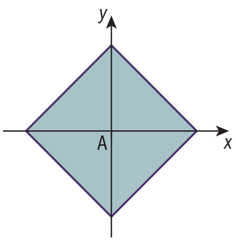
Let $z = t$; then $\begin{cases} \cancel{a} - \cancel{a}t - \cancel{a}y + \cancel{a}y + dy = \cancel{a} + 3d - \cancel{a} \\ x = 1 - t - y \end{cases}$

$$\Rightarrow \begin{cases} y = 3 - 2t \\ x = -2 + t \end{cases}$$

$$\therefore L: \vec{r} = \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

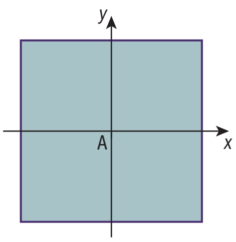
Exercise 4

a



The ‘circle’ has the shape of a square with center at A and diagonal of length $2r$.

b



The ‘circle’ has the shape of a square with center at A and side of length $2r$.

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