3 The long journey of mathematics

Try this worksheet after you have completed Exercise 3I.

The proof of the rational zero theorem and the proposition

Given a polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$, $a_i \in \mathbb{Z}$, $a_n \neq 0$ and a rational number $\frac{p}{q}$, gcd $(p, q) = 1$ in its simplest form such that $f\left(\frac{p}{q}\right) = 0$, then p is a factor of a_0 and q is a factor of a_n.

Proof

$f\left(\frac{p}{q}\right) = a_n\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \ldots + a_2\left(\frac{p}{q}\right)^2 + a_1\frac{p}{q} + a_0 = 0$. When we multiply the equation by q^n we obtain the following: $a_n p^n + a_{n-1}p^{n-1}q + \ldots + a_2p^2q^{n-2} + a_1pq^{n-1} + a_0q^n = 0$.

Rearranging the equation we can get $p\,(a_np^{n-1} + a_{n-1}p^{n-2}q + \ldots + a_2pq^{n-2} + a_1q^{n-1}) = -a_0q^n$. Since the right-hand side has a factor p and gcd $(p, q) = 1$ then we can conclude that p is a factor of a_0.

In a similar way, if we rearrange the same equation to obtain $a_np^n = -q(a_{n-1}p^{n-1} + \ldots + a_2p^2q^{n-3} + a_1pq^{n-2} + a_0q^{n-1})$. Again, since the right-hand side has a factor q and gcd $(p, q) = 1$ then we can conclude that q is a factor of a_n. QED

Proposition 2

Given a polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$, $a_i \in \mathbb{Z}$, $a_n \neq 0$ and a rational number $\frac{p}{q}$, gcd$(p, q) = 1$ such that $f\left(\frac{p}{q}\right) = 0$, then for any real value k, $(p - qk)$ is a factor of $f(k)$.

Proof

$$f\left(\frac{p}{q}\right) = a_n\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \ldots + a_2\left(\frac{p}{q}\right)^2 + a_1\frac{p}{q} + a_0 = 0$$

$$f(k) = a_nk^n + a_{n-1}k^{n-1} + \ldots + a_2k^2 + a_1k + a_0$$

When we subtract these two equations we get

$$-f(k) = a_n\left(\left(\frac{p}{q}\right)^n - k^n\right) + a_{n-1}\left(\left(\frac{p}{q}\right)^{n-1} - k^{n-1}\right) + \ldots + a_2\left(\left(\frac{p}{q}\right)^2 - k^2\right) + a_1\left(\frac{p}{q} - k\right).$$

Multiplying the equation by the common denominator q^n.

$$-f(k)q^n = a_n(p^n - q^nk^n) + a_{n-1}q(p^{n-1} - q^{n-1}k^{n-1}) + \ldots + a_2q^{n-2}(p^2 - q^2k^2) + a_1q^{n-1}(p - qk).$$

Since the terms on the right-hand side of the equation are grouped in such a way that every term of the form $(p^r - k^rq^r)$, $r = 1, 2, \ldots, n$, has a factor $(p - kq)$. Now since $(p - qk)$ and q have no common factor, $(p - qk)$ is a factor of $f(k)$. QED

Solving systems of linear equations by the method of determinants

Now let's take two linear equations in a general form.

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$$

The solution can be written in a general form using direct formulas

$$(x, y) = \left(\frac{ed - fb}{ad - bc}, \frac{af - ec}{ad - bc}\right), ad - bc \neq 0$$

Gabriel Cramer (1704–1752) developed this formula within his work on determinants.

The form $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is called a determinant of the order 2.

It is a numerical value that is calculated in the following way $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

It can be seen that all of the expressions in the formulae above can be written as determinants.

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = D$, $\begin{vmatrix} e & b \\ f & d \end{vmatrix} = D_x \begin{vmatrix} a & e \\ c & f \end{vmatrix} = D_y$, so to calculate the values of x and y we need to calculate those three determinants and then we use the formulae $x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$, $D \neq 0$.

Cramer's method can also be used for solving systems of three equations by three unknowns.

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

Then the solutions can be written as $x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$, $z = \frac{D_z}{D}$, $D \neq 0$ where

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{ and } D = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

To find the numerical value of those determinants we can use Sarrus' rule.

Pierre Frederic Sarrus (1798–1861) developed a simple rule for expanding determinants of the order 3.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{matrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{matrix} = aei + bfg + cdh - ceg - afh - bdi$$

EXAMPLE 1

Solve the simultaneous equations from Example 43 by the method of determinants.

$$\begin{cases} 2x + 4y + z = 5 \\ 3x - 5y - z = 4 \\ x + y - z = 6 \end{cases}$$

Answer

$$D = \begin{vmatrix} 2 & 4 & 1 \\ 3 & -5 & -1 \\ 1 & 1 & -1 \end{vmatrix} = \begin{matrix} 2 & 4 & 1 & 2 & 4 \\ 3 & -5 & -1 & 3 & -5 \\ 1 & 1 & -1 & 1 & 1 \end{matrix} = 10 - 4 + 3 + 5 + 2 + 12 = 28$$

$$D_x = \begin{vmatrix} 5 & 4 & 1 \\ 4 & -5 & -1 \\ 6 & 1 & -1 \end{vmatrix} = \begin{matrix} 5 & 4 & 1 & 5 & 4 \\ 4 & -5 & -1 & 4 & -5 \\ 6 & 1 & -1 & 6 & 1 \end{matrix} = 25 - 24 + 4 + 30 + 5 + 16 = 56$$

$$D_y = \begin{vmatrix} 2 & 5 & 1 \\ 3 & 4 & -1 \\ 1 & 6 & -1 \end{vmatrix} = \begin{matrix} 2 & 5 & 1 & 2 & 5 \\ 3 & 4 & -1 & 3 & 4 \\ 1 & 6 & -1 & 1 & 6 \end{matrix} = -8 - 5 + 18 - 4 + 12 + 15 = 28$$

$$D_z = \begin{vmatrix} 2 & 4 & 5 \\ 3 & -5 & 4 \\ 1 & 1 & 6 \end{vmatrix} = \begin{matrix} 2 & 4 & 5 & 2 & 4 \\ 3 & -5 & 4 & 3 & -5 \\ 1 & 1 & 6 & 1 & 1 \end{matrix} = -60 + 16 + 15 + 25 - 8 - 72 = -84$$

So the solution is

$x = \frac{56}{28} = 2$, $y = \frac{28}{28} = 1$, $z = \frac{-84}{28} = -3$

Determinants can be used with complex numbers too.

Special cases of nonlinear simultaneous equations

If we take Viète's formulae from section 3.4 we can solve some special cases of simultaneous nonlinear equations.

EXAMPLE 2

Solve the simultaneous equations

$$x + y + z = 4$$
$$xy + yz + xz = -7$$
$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{7}{10}$$

Answer

Let's take a polynomial $f(x) = x^3 + ax^2 + bx + c$ whose zeroes are x, y and z.

$x + y + z = 4 \Rightarrow a = -4$

$xy + yz + xz = -7 \Rightarrow b = -7$

$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{7}{10} \Rightarrow \frac{yz+xz+xy}{xyz} = \frac{7}{10}$

$\Rightarrow \frac{b}{-c} = \frac{7}{10} \Rightarrow c = 10$

$f(x) = x^3 - 4x^2 - 7x + 10$

Possible zeroes: $\{\pm 1, \pm 2, \pm 5, \pm 10\}$

	1	-4	-7	10
		+	+	+
1 ·		1	-3	-10
	$\boxed{1}$	$\boxed{-3}$	$\boxed{-10}$	$\boxed{0}$
		+	+	
5 ·		5	10	
	$\boxed{1}$	$\boxed{2}$	$\boxed{0}$	

Notice that due to the symmetrical form of the equations any of the values x, y and z could be any of the zeroes of the polynomial f, therefore we have six possible triplet solutions satisfying the given system.

$f(x) = (x - 1)(x - 5)(x + 2)$

$f(x) = 0 \Rightarrow x_1 = 1,\ x_2 = 5,\ x_3 = -2$

$(x, y, z) \in \{(1, 5, -2), (1, -2, 5), (5, 1, -2), (5, -2, 1), (-2, 1, 5), (-2, 5, 1)\}$

Exercise 1

1 Use Viète's formulae to solve the following simultaneous equations:

a $$\begin{cases} x + y + z = \frac{13}{2} \\ xy + yz + xz = 11 \\ xyz = 4 \end{cases}$$

b $$\begin{cases} x + y + z = 3 \\ xy + yz + xz = -13 \\ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{13}{15} \end{cases}$$

c $$\begin{cases} x^2 + y^2 + z^2 = 14 \\ xyz = -6 \\ x + y + z = 0 \end{cases}$$

d $$\begin{cases} x + y + z + w = 8 \\ xy + xz + xw + yz + yw + zw = 21 \\ xyz + xyw + xzw + yzw = 22 \\ xyzw = 8 \end{cases}$$

Chapter 3 extension worked solutions

Exercise 1

1 **a** Let's take a polynomial $f(x) = x^3 + ax^2 + bx + c$ whose zeroes are x, y and z.

$$\begin{cases} x + y + z = \frac{13}{2} \Rightarrow a = -\frac{13}{2} \\ xy + yz + xz = 11 \Rightarrow b = 11 \\ xyz = 4 \Rightarrow c = -4 \end{cases}$$

Therefore the polynomial is $f(x) = x^3 - \frac{13}{2}x^2 + 11x - 4$.

The polynomial with the same zeroes is $f_2(x) = 2x^3 - 13x^2 + 22x - 8$.
Possible integer zeroes: $\{\pm 1, \pm 2, \pm 4, \pm 8\}$

	2	-13	22	-8
		+	+	+
2 ·		4	-18	8
	2	-9	4	0
		+	+	
4 ·		8	-4	
	2	-1	0	

$f_2(x) = (x - 2)(x - 4)(2x - 1)$

$f(x) = 0 \Rightarrow x_1 = 2,\ x_2 = 4,\ x_3 = \frac{1}{2}$

$(x, y, z) \in \left\{\left(\frac{1}{2}, 2, 4\right), \left(\frac{1}{2}, 4, 2\right), \left(2, \frac{1}{2}, 4\right), \left(2, 4, \frac{1}{2}\right), \left(4, \frac{1}{2}, 2\right), \left(4, 2, \frac{1}{2}\right)\right\}$

b Let's take a polynomial $f(x) = x^3 + ax^2 + bx + c$ whose zeroes are x, y and z.

$x + y + z = 3 \Rightarrow a = -3$
$xy + yz + xz = -13 \Rightarrow b = -13$
$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{13}{15} \Rightarrow \frac{yz + xz + xy}{xyz} = \frac{13}{15}$
$\Rightarrow \frac{b}{-c} = \frac{13}{15} \Rightarrow c = 15$
$f(x) = x^3 - 3x^2 - 13x + 15$
Possible zeroes: $\{\pm 1, \pm 3, \pm 5, \pm 15\}$

	1	-3	-13	15
		+	+	+
1 ·		1	-2	-15
	1	-2	-15	0
		+	+	
5 ·		5	15	
	1	3	0	

$f(x) = (x - 1)(x - 5)(x + 3)$

$f(x) = 0 \Rightarrow x_1 = 1,\ x_2 = 5,\ x_3 = -3$

$(x, y, z) \in \{(1, 5, -3), (1, -3, 5), (5, 1, -3), (5, -3, 1), (-3, 1, 5), (-3, 5, 1)\}$

c Let's take a polynomial $f(x) = x^3 + ax^2 + bx + c$ whose zeroes are x, y and z.

$x + y + z = 0 \Rightarrow a = 0$

$xyz = -6 \Rightarrow c = 6$

$x^2 + y^2 + z^2\ 14 \Rightarrow (x + y + z)^2 - 2(xy + xz + yz) = 14$

$\Rightarrow (-a)^2 - 2b = 14 \Rightarrow 0 - 2b = 14 \Rightarrow b = -7$

$f(x) = x^3 - 7x + 6$

Possible zeroes: $\{\pm 1, \pm 2, \pm 3, \pm 6\}$

	1	0	-7	6
		+	+	+
1 ·		1	1	-6
	1	1	-6	0
		+	+	
2 ·		2	6	
	1	3	0	

$f(x) = (x - 1)(x - 2)(x + 3)$

$f(x) = 0 \Rightarrow x_1 = 1, x_2 = 2, x_3 = -3$

$(x, y, z) \in \{(1, 2, -3), (1, -3, 2), (2, 1, -3), (2, -3, 1), (-3, 1, 2), (-3, 2, 1)\}$

d Let's take a polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$ whose zeroes are x, y, z and w.

$x + y + z + w = 8 \Rightarrow a = -8$

$xy + xz + xw + yz + yw + zw = 21 \Rightarrow b = 21$

$xyz + xyw + xzw + yzw = 22 \Rightarrow c = -22$

$xyzw = 8 \Rightarrow d = 8$

$f(x) = x^4 - 8x^3 + 21x^2 - 22x + 8$

Possible zeroes: $\{\pm 1, \pm 2, \pm 4, \pm 8\}$

	1	-8	21	-22	8
		+	+	+	+
1 ·		1	-7	14	-8
	1	-7	14	-8	0
		+	+	+	
1 ·		1	-6	8	
	1	-6	8	0	
		+	+		
2 ·		2	-8		
	1	-4	0		

$f(x) = (x - 1)^2 (x - 2)(x - 4)$

$f(x) = 0 \Rightarrow x_1 = 1, x_2 = 1, x_3 = 2, x_4 = 4$

$(x, y, z, w) \in \{(1, 1, 2, 4), (1, 1, 4, 2), (1, 2, 1, 4), (1, 2, 4, 1), (1, 4, 1, 2), (1, 4, 2, 1),$
$(2, 1, 1, 4), (2, 1, 4, 1), (2, 4, 1, 1) (4, 1, 1, 2), (4, 1, 2, 1), (4, 2, 1, 1)\}$