2 Mathematics as a language

Try this worksheet after you have completed section 2.4

Investigation – order of transformations

1 As seen at the end of the chapter, you are asked to perform more than one transformation. Is there a particular order in which to perform the transformations, i.e. does order matter to the outcome?

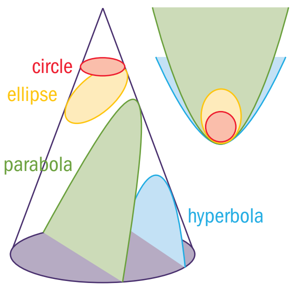
Perform the indicated transformations on $y = x^2$, and define the new function in terms of $y = f(x)$. Decide if order of transformations matters by changing the order in which you carry out the transformations below.

- **a** Shift $y = f(x)$ horizontally one unit to the left.
- **b** Reflect $y = f(x)$ in the x-axis.
- **c** Shift $y = f(x)$ vertically one unit up.
- **d** Dilate $y = f(x)$ vertically by a factor of 2.
- **e** Dilate $y = f(x)$ horizontally by a factor of 2.
- **f** Reflect $y = f(x)$ in the y-axis.

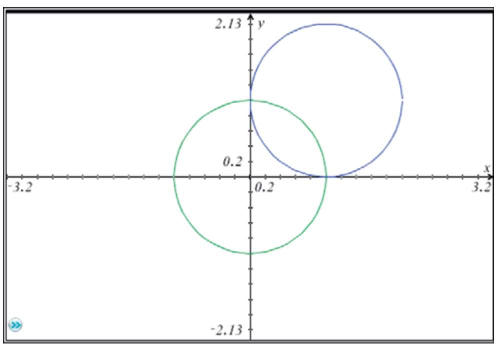
2 Given that $f(x) = x^2$, $g(x) = f\left(\frac{x+3}{2}\right)$ and $h(x) = f\left(\frac{x}{2}+3\right)$ describe the transformations that change $f(x)$ into $g(x)$ and $f(x)$ into $h(x)$. Comment on the order of transformations, and conjecture if a desired outcome necessitates a certain order of transformations.

Conic sections and transformations

Conic sections are obtained by slicing a cone with a plane, as seen in the picture.
We will focus on the circle and ellipse only.



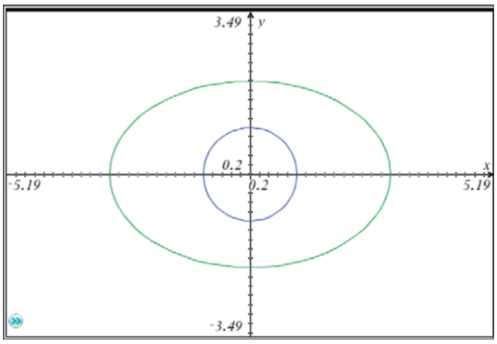
You are familiar with the equation of a circle, $x^2 + y^2 = r^2$, where r is the radius. Solving for y we obtain $y = \pm\sqrt{r^2 - x^2}$. If we let $r = 1$ we obtain $y = \sqrt{1 - x^2}$. If we now shift the graph of the circle one unit to the right and one unit upward, the new expression is $y = \pm\sqrt{1-(x-1)^2} + 1$. Bringing +1 to the left side, squaring both sides and bringing the expression in x to the left side, we obtain $(x-1)^2 + (y-1)^2 = 1$ Graphing both we see that the transformation is a circle whose center is at $(1, 1)$.



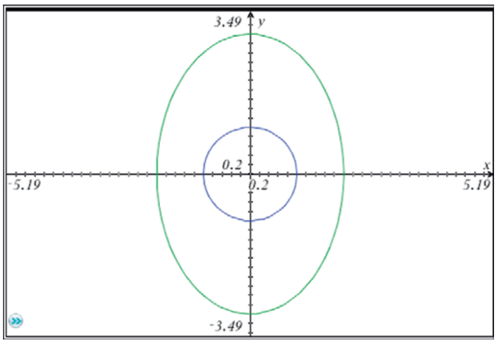
Hence, the equation of a circle whose center is not at the origin, but rather at (h, k) is $(x - h)^2 + (y - k)^2 = r^2$, where h is the horizontal shift and k is the vertical shift of a circle whose center is the origin with radius r.

We will now dilate the unit circle horizontally by a factor of 3 and vertically by a factor of $\frac{1}{2}$. We then obtain $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$. Solving for y, $y = \pm 2\sqrt{1 - \left(\frac{x}{3}\right)^2}$

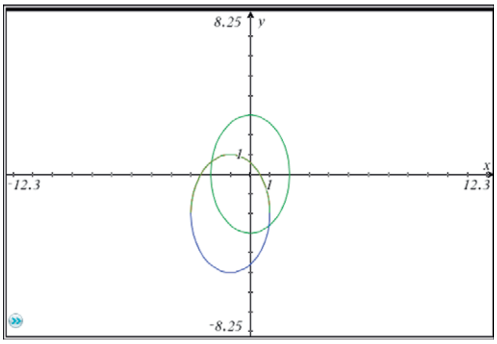
Graphing both we obtain the following graph. The figure formed is called an ellipse, with the major axis being the x-axis.



If we now dilate the unit circle horizontally by a factor of 2 and vertically by a factor of 3, we obtain $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$. Solving for y and graphing, we obtain the following graph, where the major axis is the y-axis.



If we now perform a vertical and horizontal shift on the dilated ellipse above, e.g., a vertical shift of 2 units down and a horizontal shift of 1 unit to the left, we obtain $\left(\frac{x+1}{2}\right)^2 + \left(\frac{y-2}{3}\right)^2 = 1$, where the center of the ellipse is $(-1, 2)$.



As you can see, an ellipse is of the form $\left(\frac{x-h}{a}\right)^2 + \left(\frac{y-k}{b}\right)^2 = 1$, whose center is (h, k). It is a circle that has been transformed by a horizontal dilation of factor a, and a vertical dilation of factor b, where $a, b \in \mathbb{R}^+$. If $a > b$, then the major axis is the x-axis; if $b > a$ then the major axis is the y-axis.

Exercise 1

1 Define the ellipse that is determined by the following transformations on the unit circle.

a A vertical dilation of factor 5, and a horizontal shift of two units to the right.

b A horizontal dilation of factor 3, and a vertical shift of 3 units upward.

c A horizontal dilation of factor 5, a vertical dilation of factor 2, a vertical shift 1 unit downward, and a horizontal shift 2 units to the left.

2 Define the steps necessary to transform the ellipse $\left(\frac{x+3}{4}\right)^2+\left(\frac{y-7}{6}\right)^2=1$ into the unit circle.

Chapter 2 extension worked solutions

Investigation – order of transformations

1 In general you can perform the transformations in any order you want, unless you want a particular result, as in question 2.

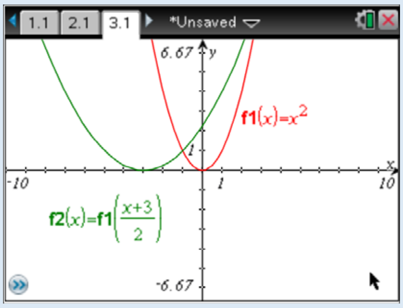
Transformations of the form $af(x) + b$ can be done in the same order as the order of arithmetic operations, i.e. dilate vertically first, then shift horizontally, but it does not really matter. Therefore, vertical dilations and vertical shifts, which are outside the 'x', can be done in any order.

Transformations of the form $af(bx)$ can be done in any order, i.e. vertical dilations followed by horizontal dilations, or vice versa.

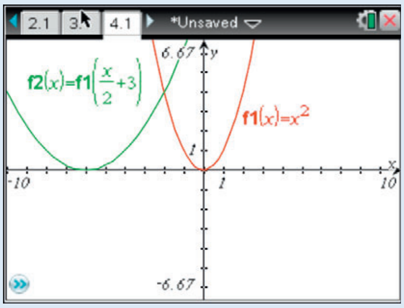
Horizontal dilations and shifts involve the argument, and hence can be more difficult. A good rule of thumb therefore is to start with the argument and work your way outward. Translations and reflections in the axes are normally affected by the order in which they are done.

2 There is a specific order of transformations to achieve a desired effect.

In order to transform $f(x) = x^2$ into $g(x) = f\left(\frac{x+3}{2}\right)$, first x has been replaced by $\frac{x}{2}$, representing a horizontal dilation of factor 2, then x has been replaced by $x + 3$, representing a horizontal shift to the left of 3 units. Graphing both f and g on the GDC gives this graph.



In order to transform $f(x) = x^2$ into $h(x) = f\left(\frac{x}{2} + 3\right)$, first we replace x with $x + 3$, then replace x with $\frac{x}{2}$. Combining into a single fraction we obtain $f\left(\frac{x}{2} + 3\right) = f\left(\frac{x+6}{2}\right)$, i.e. a horizontal translation of 6 units to the left followed by a horizontal dilation of factor 2.



Exercise 1

1 **a** $(x-2)^2 + \left(\frac{y}{5}\right)^2 = 1$

b $\left(\frac{x}{3}\right)^2 + (y+3)^3 = 1$

c $\left(\frac{x+2}{5}\right)^2 + \left(\frac{y-1}{2}\right)^2 = 1$

2 A horizontal shift of 3 units to the right, a vertical shift of 7 units upward, a horizontal dilation of factor $\frac{1}{4}$ and a vertical dilation of factor $\frac{1}{6}$.