12 Multiple perspective in mathematics

Exercise

1 Given two non-zero complex numbers $z_1 = r_1\cos\theta_1 + \mathrm{i}\,r_1\sin\theta_1$ and $z_2 = r_2\cos\theta_2 + \mathrm{i}\,r_2\sin\theta_2$ show that $|z_1 + z_2| \le |z_1| + |z_2|$. State the geometrical meaning of this inequality in terms of the vectors that represent the complex numbers z_1, z_2 and $z_1 + z_2$. Hence, write down the condition for which the relation $|z_1 + z_2| = |z_1| + |z_2|$ holds.

Try this question after you have completed Exercise 12B

2 Let $z = \mathrm{cis}\alpha$. Show that $z - 1 = 2\mathrm{i}\sin\frac{\alpha}{2}\,\mathrm{cis}\frac{\alpha}{2}$. Hence show that $\left(\frac{z-1}{z+1}\right)^2 \equiv -\tan^2\frac{\alpha}{2}$.

3 Given $z = \mathrm{cis}\theta$, show that $\frac{2}{1-z} \equiv 1 + \mathrm{i}\cot\frac{\theta}{2}$.

Try these questions after you have completed Exercise 12D

4 Use mathematical induction to show that $\left(1 + \mathrm{i}\sqrt{3}\right)^n = 2^n\,\mathrm{cis}\left(\frac{n\pi}{3}\right)$, for any $n \in \mathbb{Z}^+$.

Explain how to extend this result to all values of $n \in \mathbb{Z}$. State any assumptions you need to make.

Try this question after you have completed Exercise 12E

5 Solve the equations:

a $z^3 = (\mathrm{i} - z)^3$

b $(1 + \mathrm{i}z)^n = (1 - \mathrm{i}z)^n$

Hence compare the number of solutions to part **b** for n odd and n even.

6 Give a geometrical argument that explains why the set of the $2n$th roots of the unity is equal to the set of the nth roots of the unity together with their negatives.

7 Consider the points $A_1, A_2, \ldots, A_n$ equally spaced around a unit circle.

Prove that $A_1A_2 \times A_1A_3 \times \ldots A_1A_n = n$. Hence show that $\left|\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ldots\sin\frac{(n-1)\pi}{n}\right| = \frac{n}{2^{n-1}}$.

Try these questions after you have completed Exercise 12F

8 Show that $\tan 6\alpha \equiv \frac{6 - 20\tan^2\alpha + 6\tan^4\alpha}{1 - 15\tan^2\alpha + 15\tan^4\alpha - \tan^6\alpha}$.

9 Use the results $z^n + \left(\frac{1}{z}\right)^n = 2\cos(n\alpha)$ and $z^n - \left(\frac{1}{z}\right)^n = 2\mathrm{i}\sin(n\alpha)$ to show that

$\cos^4\alpha + \sin^4\alpha \equiv \frac{3 + \cos 4\alpha}{4}$.

Try these questions after you have completed Exercise 12G

Chapter 12 extension worked solutions

1 $z_1 = r_1 \operatorname{cis}\theta_1 = r_1\cos\theta_1 + \mathrm{i}r_1\sin\theta_1$

$z_2 = r_2 \operatorname{cis}\theta_2 = r_2\cos\theta_2 + \mathrm{i}r_2\sin\theta_2$

$z_1 + z_2 = (r_1\cos\theta_1 + r_2\cos\theta_2) + \mathrm{i}(r_1\sin\theta_1 + r_2\sin\theta_2)$

$|z_1 + z_2| = \sqrt{(r_1\cos\theta_1 + r_2\cos\theta_2)^2 + \mathrm{i}(r_1\sin\theta_1 + r_2\sin\theta_2)^2}$

$$= \sqrt{r_1^2 + r_2^2 + 2r_1r_2\underbrace{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)}_{\cos(\theta_1-\theta_2)}} \le \sqrt{r_1^2 + r_2^2 + 2r_1r_2}$$

As $\cos(\theta_1 - \theta_2) \le 1 \le \sqrt{(r_1 + r_2)^2}$

$= r_1 + r_2 = |z_1| + |z_1|$

Geometrically this means that the length of the vector that corresponds to $z_1 + z_2$ is less or equal to the sum of the lengths of the vectors that represent z_1 and z_2. This property is called triangular inequality. The equality occurs when the three vectors have the same direction.

2 $z = \operatorname{cis}\alpha \Rightarrow z = \cos\alpha + \mathrm{i}\sin\alpha$

$\Rightarrow z - 1 = (\cos\alpha - 1) + \mathrm{i}\sin\alpha$

$\Rightarrow z - 1 = \left(\cancel{1} - 2\sin^2\frac{\alpha}{2} - \cancel{1}\right) + \mathrm{i}\left(2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right)$

$\Rightarrow z - 1 = 2\sin\frac{\alpha}{2}\left(-\sin\frac{\alpha}{2} + \mathrm{i}\cos\frac{\alpha}{2}\right)$

$\mathrm{i}^2 = -1$ so

$z - 1 = 2\mathrm{i}\sin\frac{\alpha}{2}\left(\cos\frac{\alpha}{2} + \mathrm{i}\sin\frac{\alpha}{2}\right)$

$\therefore z - 1 = 2\mathrm{i}\sin\frac{\alpha}{2}\operatorname{cis}\left(\frac{\alpha}{2}\right)$

$$\left.\begin{aligned} z + 1 &= (\cos\alpha + 1) + \mathrm{i}\sin\alpha \\ &= 2\cos^2\frac{\alpha}{2} + 2\mathrm{i}\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} \\ &= 2\cos\frac{\alpha}{2}\operatorname{cis}\frac{\alpha}{2} \end{aligned}\right\} \text{following the same method as above}$$

$$\therefore \left(\frac{z-1}{z+1}\right)^2 = \left(\frac{2\mathrm{i}\sin\frac{\alpha}{2}\cancel{\operatorname{cis}\frac{\alpha}{2}}}{2\cos\frac{\alpha}{2}\cancel{\operatorname{cis}\frac{\alpha}{2}}}\right)^2$$

$$= \left(\mathrm{i}\tan\frac{\alpha}{2}\right)^2 = -\tan^2\frac{\alpha}{2},\ \alpha \ne k\pi, k \in \mathbb{Z}$$

$$\therefore \left(\frac{z-1}{z+1}\right)^2 \equiv -\tan^2\frac{\alpha}{2}$$

Given $z = \operatorname{cis}\theta$, show that $\frac{2}{1-z} \equiv 1 + \mathrm{i}\cot\frac{\theta}{2}$.

3 $z = \cos\theta + \mathrm{i}\sin\theta, \quad z \neq 1$

$$\Rightarrow \frac{2}{z-1} = \frac{2}{(1-\cos\theta) + \mathrm{i}\sin\theta} = \frac{2(1-\cos\theta) + 2\mathrm{i}\sin\theta}{(1-\cos\theta)^2 + \sin^2\theta}.$$

As

$$\cos\theta = 2\cos^2\frac{\theta}{2} - 1$$

and

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2},$$

$$1 - \cos\theta = 1 - 2\cos^2\frac{\theta}{2} + 1 = 2\left(1 - \cos^2\frac{\theta}{2}\right) = 2\sin^2\frac{\theta}{2}.$$

$$\therefore \frac{2}{1-z} = \frac{4\sin^2\frac{\theta}{2} + 4\mathrm{i}\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{4\sin^4\frac{\theta}{2} + 4\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}$$

$$= \frac{4\sin^2\frac{\theta}{2} + 4\mathrm{i}\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{4\sin^2\frac{\theta}{2}\left(\underbrace{\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}}_{1}\right)}$$

$$= 1 + \cot\frac{\theta}{2}, \quad \theta \neq 2k\pi, \quad k \in \mathbb{Z}$$

$$\therefore \frac{2}{1-z} \equiv 1 + \cot\frac{\theta}{2}.$$

4 Let $P(n)$: $\left(1 + \mathrm{i}\sqrt{3}\right)^n = 2^n \operatorname{cis}\left(\frac{n\pi}{3}\right), \quad n \in \mathbb{Z}^+$

Step 1

$P(1)$: $\left(1+\mathrm{i}\sqrt{3}\right)^1 = 2^1\operatorname{cis}\left(\frac{\pi}{3}\right)$

$$\Rightarrow 1 + \mathrm{i}\sqrt{3} = 2(\underbrace{\cos\tfrac{\pi}{3}}_{\frac{1}{2}} + \mathrm{i}\underbrace{\sin\tfrac{\pi}{3}}_{\frac{\sqrt{3}}{2}})$$

$\therefore$ $P(1)$ is true

Step 2 Assume that $P(k)$ is true

$$\left(1 + \mathrm{i}\sqrt{3}\right)^k = 2^k\operatorname{cis}\left(\frac{k\pi}{3}\right)$$

Consider

$$P(k+1): \underbrace{\left(1 + \mathrm{i}\sqrt{3}\right)^{k+1}}_{\text{LHS}} = \underbrace{2^{k+1}\operatorname{cis}\left(\frac{(k+1)\pi}{3}\right)}_{\text{RHS}}$$

As

$$\text{LHS} = \left(1 + \mathrm{i}\sqrt{3}\right)^{k+1} = \left(1 + \mathrm{i}\sqrt{3}\right)^k\left(1 + \mathrm{i}\sqrt{3}\right)$$

$$= \left(2^k\operatorname{cis}\left(\frac{k\pi}{3}\right)\right)\left(2\operatorname{cis}\left(\frac{\pi}{3}\right)\right)$$

$$= 2^k \times 2\operatorname{cis}\left(\frac{k\pi}{3} + \frac{\pi}{3}\right)$$

$$= 2^{k+1} 2\operatorname{cis}\left(\frac{(k+1)\pi}{3}\right) = \text{RHS}$$

$P(k)$ true $\Rightarrow P(k+1)$ true

Therefore, as $P(1)$ is true, by the principle of mathematical induction, we can conclude that $P(n)$ is true for all values of $n \in \mathbb{Z}^+$.

If we consider that $z^0 = 1$ and $z^{-n} = \left(\frac{1}{2}\right)^n = \frac{1}{z^n}$ we can extend the result to $n \in \mathbb{Z}$.

1 $\underbrace{\left(1 + \mathrm{i}\sqrt{3}\right)^0}_{1} = \underbrace{2^0}_{1}\,\underbrace{\operatorname{cis}(0)}_{1} \quad \therefore P(0)$ true

2 $\left(1 + \mathrm{i}\sqrt{3}\right)^{-n} = \dfrac{1}{\left(1 + \mathrm{i}\sqrt{3}\right)^n} = \dfrac{1}{2^n\operatorname{cis}\left(\frac{n\pi}{3}\right)} = \dfrac{1}{2^n}\operatorname{cis}\left(-\dfrac{n\pi}{3}\right)$

$$= 2^{-n}\operatorname{cis}\left(\frac{(-n)\pi}{3}\right), \quad n \in \mathbb{Z}^+$$

$\therefore$ the result is valid for all $n \in \mathbb{Z}^+$

5 **a** $z^3 = (i - z)^3 \Rightarrow \left(\frac{z}{i-z}\right)^3 = 1, \quad z \neq i$

$w = \frac{z}{i - z} \Rightarrow w^3 = 1$

$w^3 = \text{cis}\,0$

$w^3 = \text{cis}\left(\frac{2k\pi}{3}\right) \quad k = 0, 1, 2$

So,

$k = 0, \quad \frac{z}{i - z} = 1 \Rightarrow z^3 = i - z \Rightarrow z = \frac{i}{2}$

$k = 1, \quad \frac{z}{i - z} = \text{cis}\frac{2\pi}{3} \Rightarrow \frac{z}{i - z} = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$

$2z = (-1 + \sqrt{3}i)(i - z)$

$2z = -i + z - \sqrt{3} - i\sqrt{3}z$

$(1 + \sqrt{3}i)z = -\sqrt{3} - i$

$z = -\frac{\sqrt{3} + i}{1 + \sqrt{3}i}$

$z = -\frac{\left(\sqrt{3} + i\right)\left(1 - \sqrt{3}i\right)}{4}$

$z = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$

$k = 2, \frac{z}{i - z} = \text{cis}\left(\frac{4\pi}{3}\right) \Rightarrow \frac{z}{i - z} = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$

$\Rightarrow 2z = (-1 - \sqrt{3}i)(i - z)$

$\Rightarrow 2z = -i + z + \sqrt{3} + i\sqrt{3}z$

$\therefore (1 - \sqrt{3}i)z = \sqrt{3} - i$